Succinct Trees: Theory and Practice

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Succinct Data Structures

- Succinct data structures
  = succinct representation of data + succinct index
- Examples
  - sets
  - trees, graphs
  - strings
  - permutations, functions

Succinct Representation

- A representation of data whose size (roughly) matches the information-theoretic lower bound
- If the input is taken from L distinct ones, its information-theoretic lower bound is $\lceil \log L \rceil$ bits
- Example 1: a lower bound for a set $S \subseteq \{1,2,...,n\}$
  - $\log 2^n = n$ bits

$\emptyset$

$n = 3$

$\{1\}$ $\{2\}$ $\{3\}$

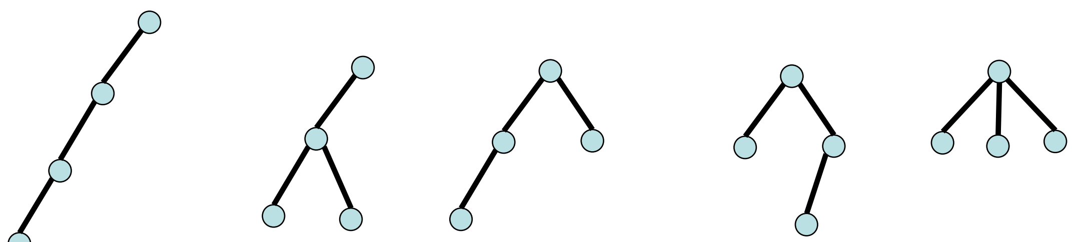
$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

$\{1,2,3\}$

Base of logarithm is 2

- Example 2: *n* node ordered tree

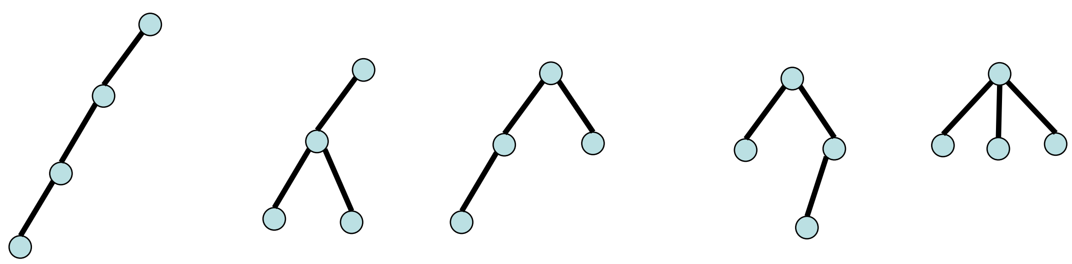
$$\log \frac{1}{2n-1}\binom{2n-1}{n-1} = 2n - \Theta(\log n)\,\text{bits}$$



$n = 4$

- For any data, there exists its succinct representation
  - enumerate all in some order, and assign codes
  - can be represented in $\lceil \log L \rceil$ bits
  - not clear if it supports efficient queries
- Query time depends on how data are represented
  - necessary to use appropriate representations

$$n = 4 \quad \left\lceil \log \frac{1}{2n-1} \binom{2n-1}{n-1} \right\rceil = \lceil \log 5 \rceil = 3 \text{ bits}$$



000 001 010 011 100

Succinct Indexes

- Auxiliary data structure to support queries
- Size: o(log L) bits
- (Almost) the same time complexity as using conventional data structures
- Computation model: word RAM
  - assume word length w = log log L (same pointer size as conventional data structures)

word RAM

- word RAM with word length w bits supports
  - reading/writing w bits of memory at arbitrary address in constant time
  - arithmetic/logical operations on two w bits numbers are done in constant time
  - arithmetic ops.: +, –, *, /, log (most significant bit)
  - logical ops.: and, or, not, shift
- These operations can be done in constant time using $O(w^\varepsilon)$ bit tables ($\varepsilon > 0$ is an arbitrary constant)

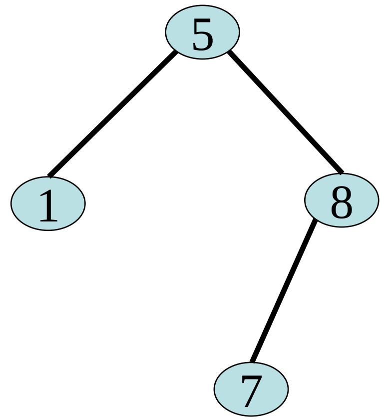
A Variety of Trees

- Ordered/Unordered trees
  - Child nodes are ordered/not ordered
  - Concerning unordered trees, trees can be regarded as the same if they become the same shape by reordering children
- Edges are labeled/not labeled
  - Node labels can be represented by edge labels on the edge toward parent nodes

- In this talk, we consider labeled ordered trees.

Applications of Ordered Trees

- An abstract data type for "dictionary"
- A data structure D is called dictionary if for a set S and a key k, it supports
  - Search(D, k): returns Yes iff $k \in S$
  - Insert(D, x): adds x to S
  - Delete(D, x): removes x from S

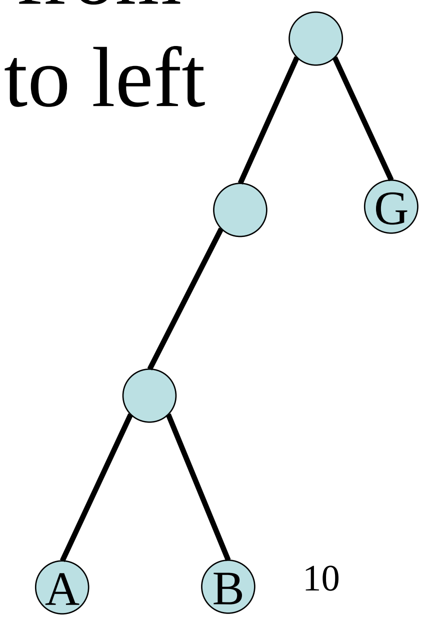


- A binary search tree is used as a dictionary.
  - Any balanced binary search tree supports the above operations in O(log n) time for a set of n elements.
  - We assume that element comparison is done in O(1) time.

Radix Search Trees

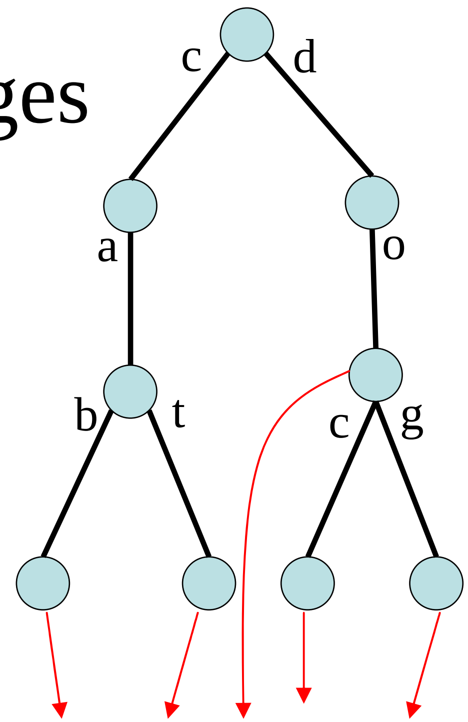
- We assume keys are *b*-bit integers
- All elements are stored in leaves
- Left (right) subtree of root stores all the keys whose first bit is 0 (1).
- To search for a key, we traverse the tree from the root. At a node with depth *d*, we go to left (right) if *d*-th bit of the key is 0 (1).
- All operations are done in O(*b*) time.

A 000
B 001
C 010
D 011
E 100
F 101
G 110
H 111



Tries

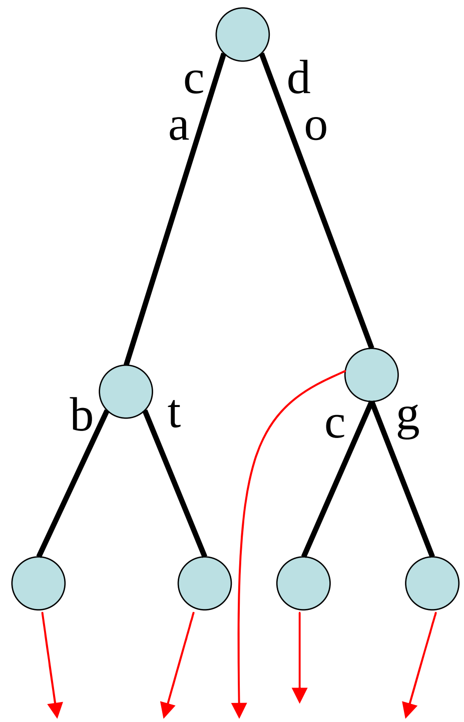
- Trie is a data structure for storing a set of strings.
- A node has at most σ children (σ: alphabet size)
- Each edge is labeled a character *c*
- The concatenation of characters on edges from the root to a node coincides with the string represented by the node



A Trie for *S* = {cab, cat, do, doc, dog}

Compressed Tries

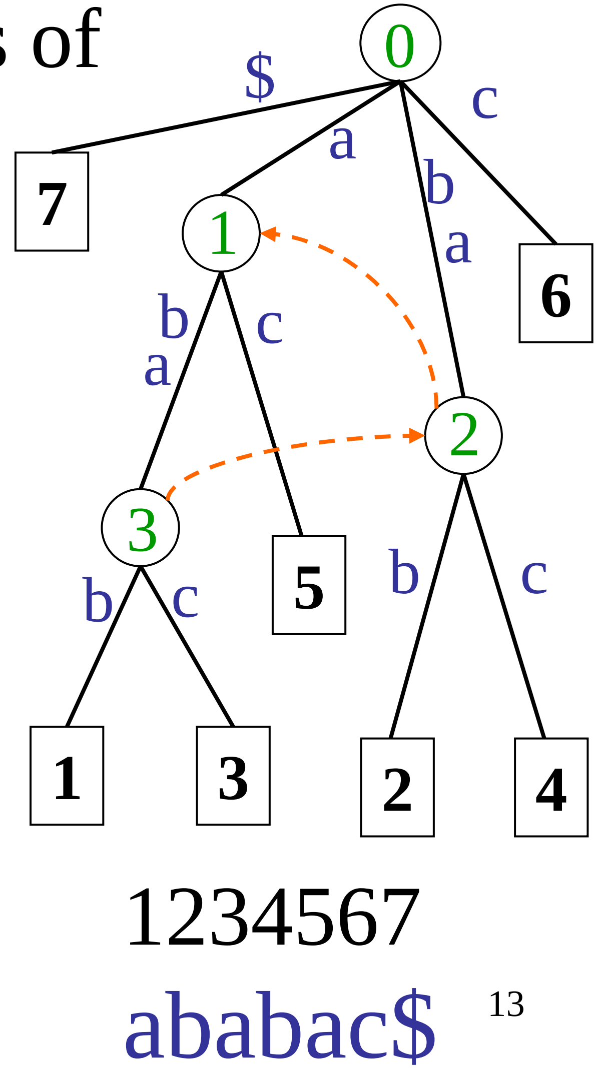
- Compressed Tries are obtained from standard Tries by compressing chains of *redundant* nodes
  - redundant node = a node with only one child
  - #nodes ≤ #leaves–1
- An edge represents a string



S = {cab, cat, do, doc, dog}

Suffix Trees

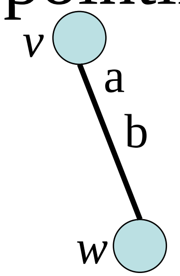
- The compressed trie for all suffixes of a string
- The suffix tree for a string consists of
  - n leaves (n: length of string)
  - $\leq n-1$ internal nodes
  - edge labels
  - node depths
  - suffix links



Operations on Trees

- Binary search trees
  - left(v), right(v): returns the left/right child node of v
  - key(v): returns the key stored in node v
- Tries
  - child(v, c): returns a child w of v with edge label c
  - key(v): returns the key stored in node v
- Compressed tries
  - child(v, c): returns a child w of v with edge label c…
  - edge(w, d): returns d-th character on edge pointing to w
  - key(v)

child(v, a) = w
edge(w, 2) = b

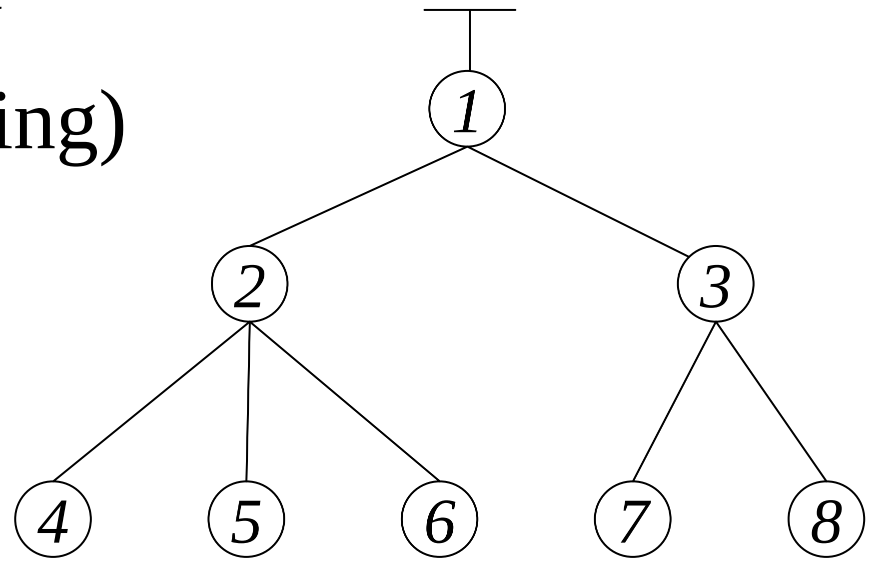


Succinct Representations of Ordered Trees

- LOUDS (level order unary degree sequence)
  - [Jacobson 89]
- BP (balanced parentheses)
  - [Munro, V. Raman 97, 01]
- DFUDS (depth first unary degree sequence)
  - [Benoit et al. 99, 05]

LOUDS Representation

- Degrees of nodes are encoded by unary codes in breadth-first order
  - degree $d \rightarrow 1^d0$
- $2n+1$ bits for n nodes (matches the lower bound)
- i-th node is represented by i-th 1 (ones-based numbering)



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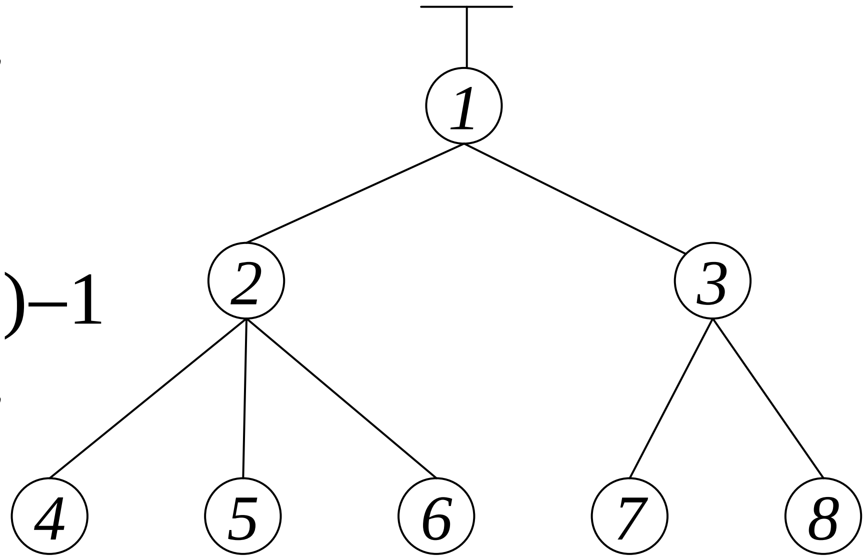
LOUDS L **10110111011000000**

Tree Navigational Operations (1)

- i-th node: $select_1(L, i)$ $(i \geq 1)$
- $firstchild(x)$
  - $y := select_0(rank_1(L,x))+1$
  - if $L[y] = 0$ then -1 else y
- $lastchild(x)$
  - $y := select_0(rank_1(L,x)+1)-1$
  - if $L[y] = 0$ then -1 else y



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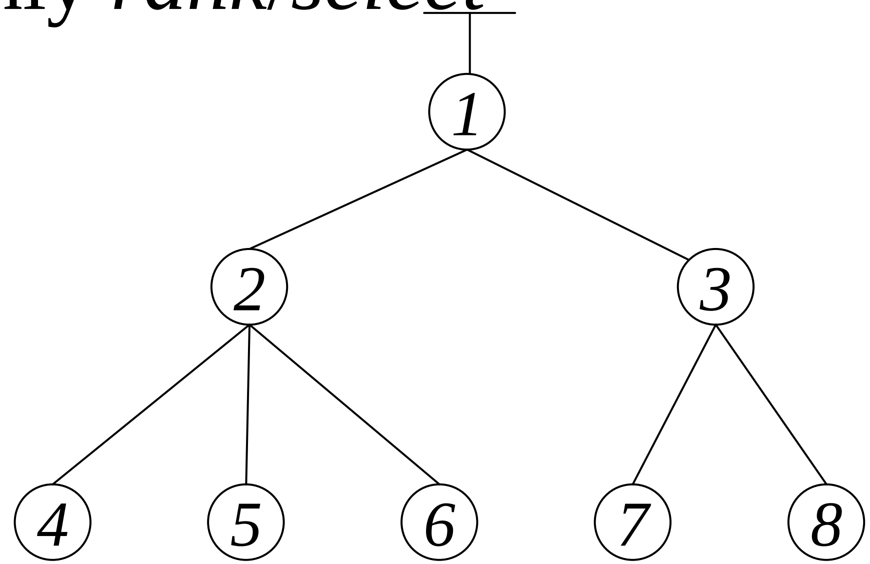
LOUDS L **10110111011000000**

Tree Navigational Operations (2)

- *sibling*(x)
  - if $L[x+1] = 0$ then -1 else $x+1$
- $parent(x) = select_1(rank_0(L,x))$
- $degree(x) = lastchild(x) - firstchild(x) + 1$
- Merits: implemented by only *rank*/*select*
- Demerits: cannot compute subtree sizes



1 2 3 4 5 6 7 8

LOUDS L **1011011101100000**

Tries using LOUDS

- child(v, c)

  $w = firstchild(v)$, $r = rank_1(L, w)$, $k = 0$

  while ($L[w+k]$!= 0) {

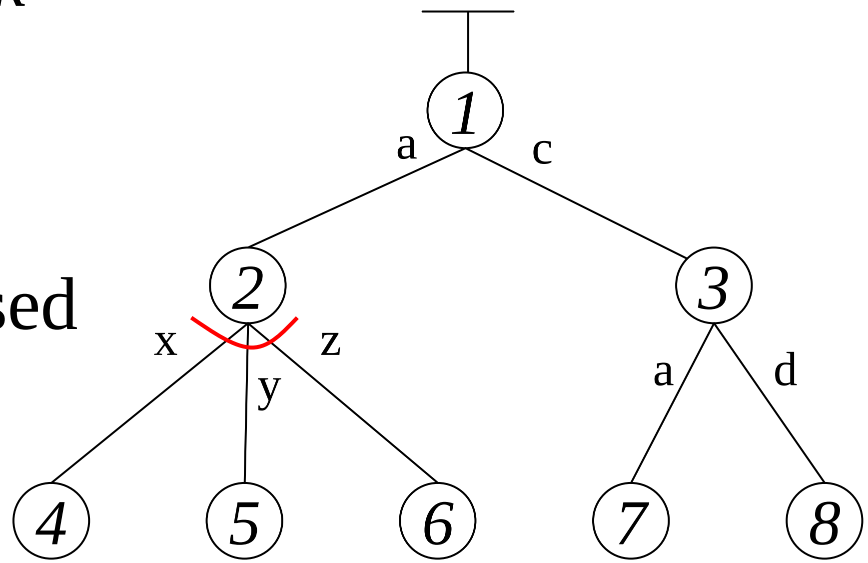
      if ($C[r+k] == c$) return $w+k$

      $k = k+1$

  }

  – Binary search can be also used

- key(v) is stored in array indexed with $rank_1(L, v)$



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LOUDS　L　**10110111011000000**

C　_ a c x y z a d

- A trie with n nodes and size-σ label set supporting child(v, c) in O(log σ) time is represented in $n(2+\log \sigma) + o(n)$ bits.
  - O(σ) time sequential search is faster for small σ.

- By using an auxiliary array, key(v) is performed in O(1) time.

Numbering of Nodes

	Ones-based	Zeros-based
i-th node	$s_1(i)$	$s_0(i+1)$
$firstchild(x)$	$s_0(r_1(x))+1$	$s_0(r_1(s_0(r_0(x)-1)+2)$
$lastchild(x)$	$s_0(r_1(x)+1)-1$	$s_0(r_1(x)+1)$
$sibling(x)$	$x+1$	$s_0(r_0(x))$
$parent(x)$	$s_1(r_0(x))$	$s_0(r_0(s_1(r_0(x)-1)+1))$

Ones-based

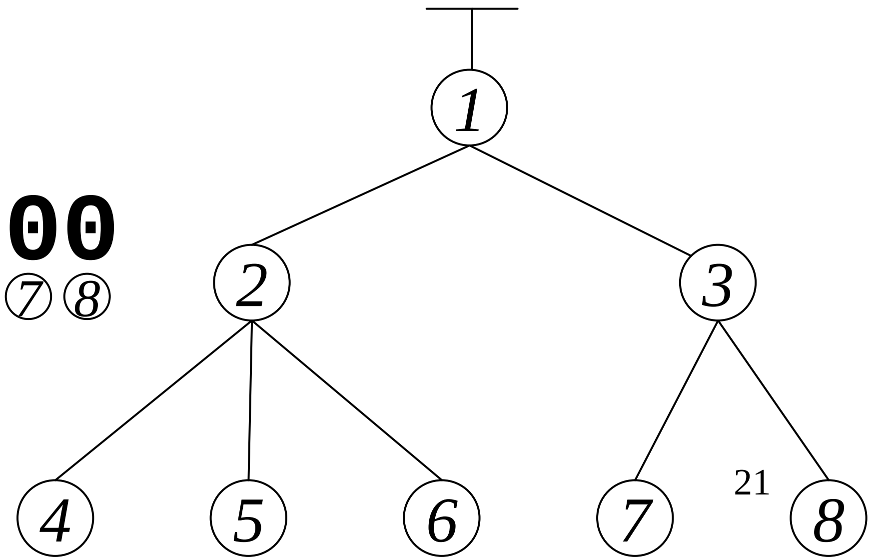
① ②③ ④⑤⑥ ⑦⑧

L **10110111011000000**

① ② ③④⑤⑥⑦⑧

Zeros-based

$s_c(i) = select_c(L, i)$, $r_c(i) = rank_c(L, i)$

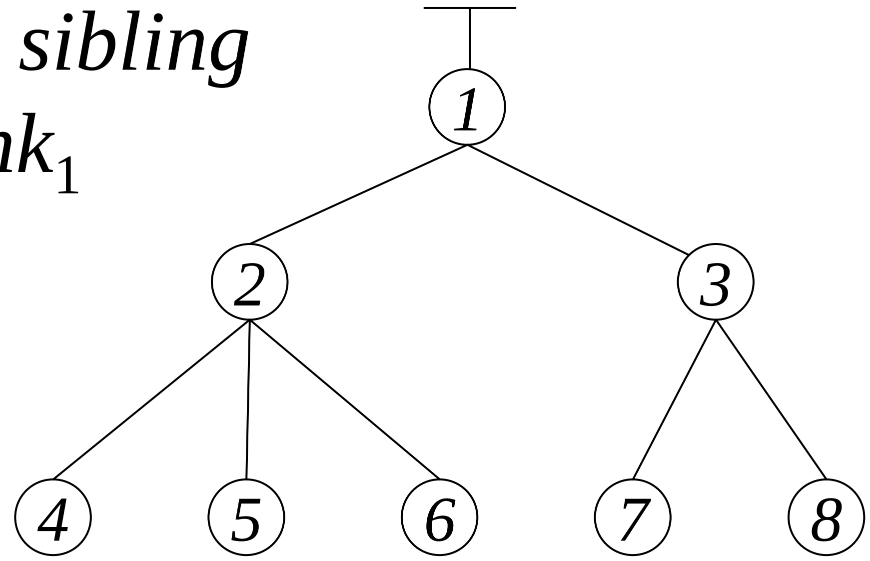


Double Numbering

- A node is represented by a pair $\langle x,y \rangle$
  - y: the level-order of the node
  - x: the position in L for the node (ones- or zeros- based)
- $parent(\langle x,y \rangle)$ is done in ones-based numbering by
  - $r = x - y$
  - $x' = s_1(r)$
  - $y' = x' - r$
  - return $\langle x',y' \rangle$
- $(parent(x) = s_1(r_0(x)))$
- The value of $r_0(x)$ can be derived from $s_1(r)$, which was already done when $\langle x,y \rangle$ was obtained.

LOUDS++

- [Delpratt, Rahman, R. Raman 06]
- LOUDS sequence L is represented by runs of ones and runs of zeros
- *parent*, *firstchild*, *lastchid*, *sibling* are done by $select_1$ and $rank_1$ on R_1 and R_0



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R_0 11100001

R_1 10100101

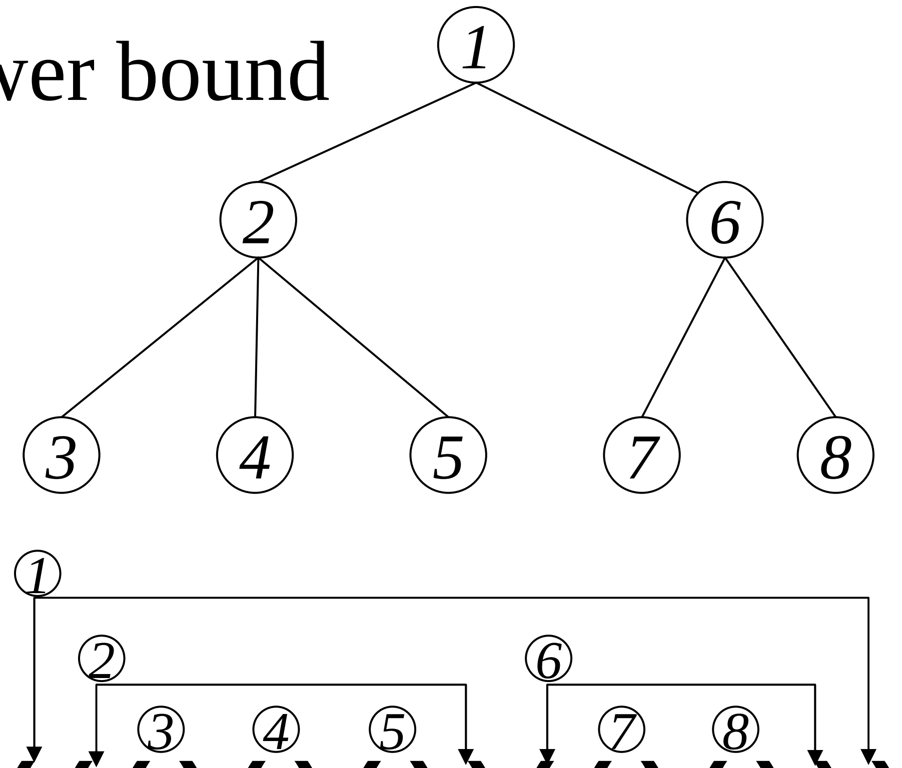
LOUDS L 101101110110000000

The Merits and Demerits of LOUDS

- Implemented by *rank* and *select*
  - easy to implement
  - fast in practice
- Suitable for labeled trees
  - child() is fast because of locality of reference
  - Tx library by Okanohara
- Many operations are not supported in O(1) time
  - subtree size
  - level ancestor
  - lowest common ancestor, etc.

BP Representation

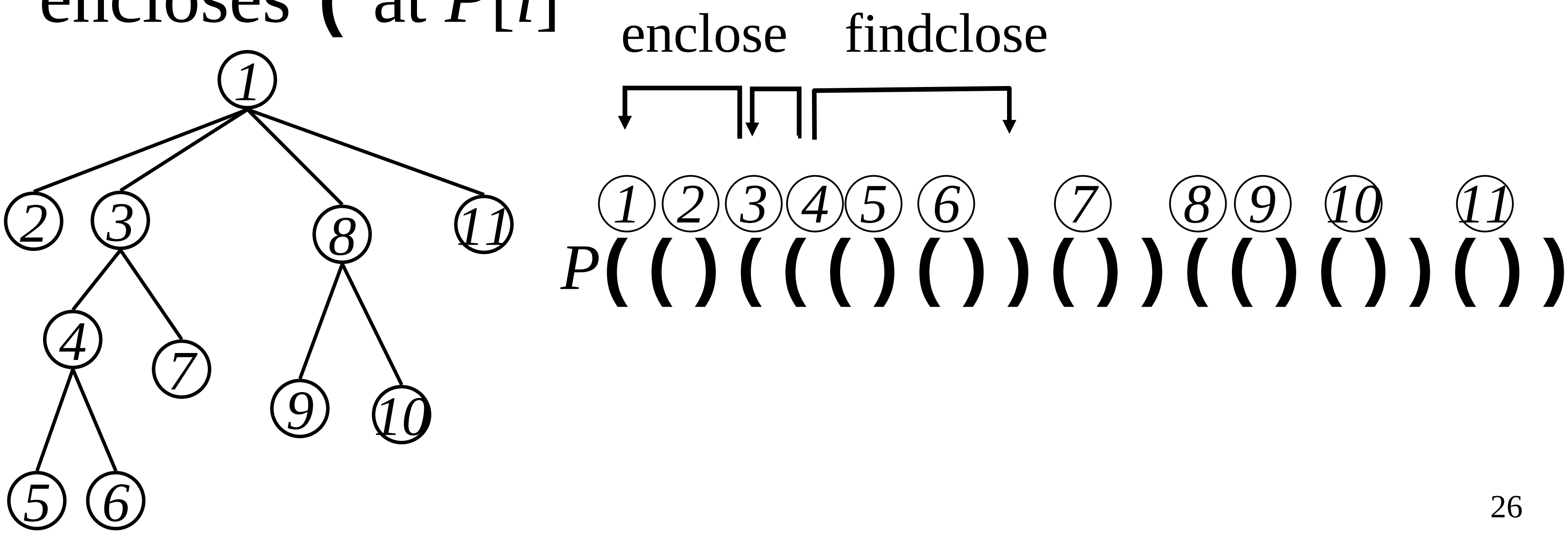
- Each node is represented by a pair of matching open and close parentheses
- 2*n* bits for *n* nodes
- The size matches the lower bound



BP P ((()()())(()()))

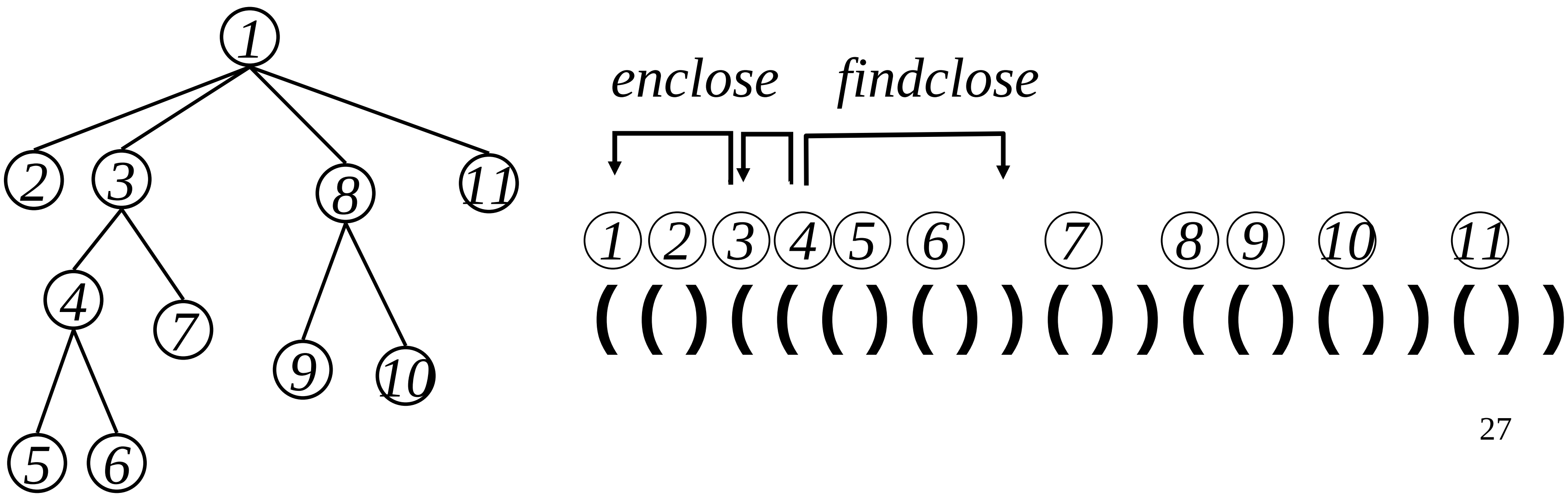
Basic Operations on BP

- A node is represented by the position of **(**
- *findclose*(P,i): returns the position of **)** matching with **(** at $P[i]$
- *enclose*(P,i): returns the position of **(** which encloses **(** at $P[i]$



Tree Navigational Operations

- *parent*(v) = *enclose*(P,v)
- *firstchild*(v) = v + 1
- *sibling*(v) = *findclose*(P,v) + 1
- *lastchild*(v) = *findopen*(P, *findclose*(P,v)–1)



Tries using BP

- child(v, c)

  w = *firstchild*(v)

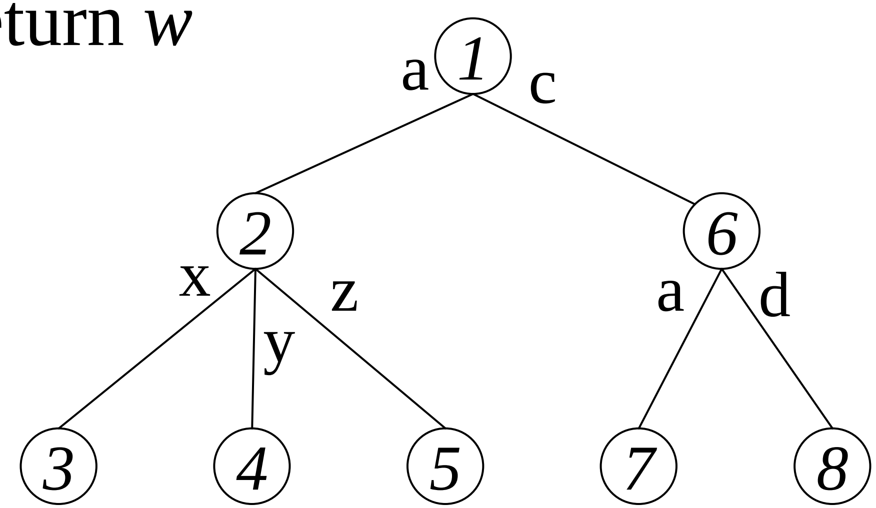
  while (w != NIL) {

      if ($C[rank_((P, w)] == c$) return w

      w = *sibling*(w)

  }

- key(v) is stored in array indexed with $rank_((P, v)$



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BP P ((( ) ( ) ( )) (( ) ( )))

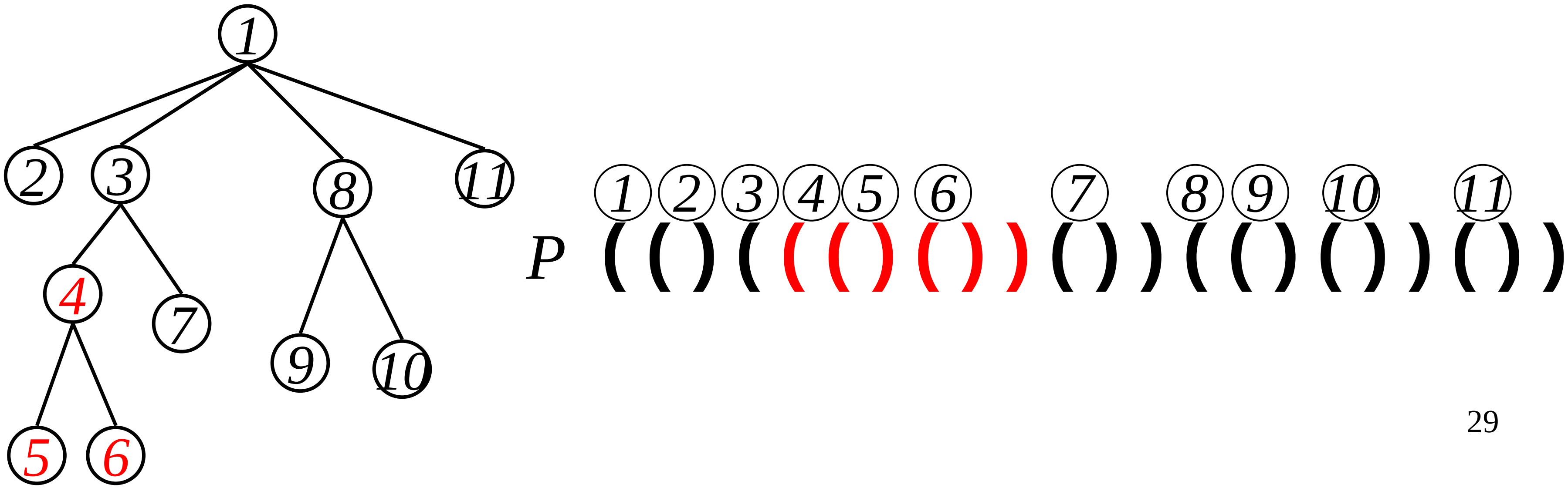
C _ a x y z c a d

Number of Descendants (Subtree Size)

- The size of the subtree rooted at *v* is

  $$subtreesize(v) = (findclose(P,v) - v + 1)/2$$

- *degree* (#children) can be computed by repeatedly applying *findclose*, but it takes time proportional to the number of children

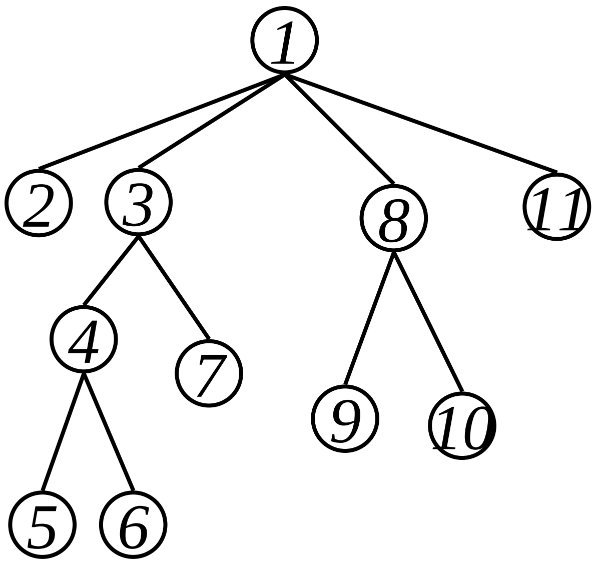


History

- [Munro, V. Raman 97, 01]
  - *firstchild*, *sibling*, *lastchild*, *parent*, *subtreesize*, *depth*
- [Munro, V. Raman, Rao 98, 01]
  - *leftmostleaf*, *rightmostleaf*, *#leaves*
- [Sadakane 02, 07]
  - *lca* (lowest common ancestor)
- [Munro, Rao 04]
  - level ancestor
- [Geary, Rahman, R. Raman, V. Raman 04, 06]
  - simpler *findopen* and *enclose*
- [Lu, Yeh 08]
  - *degree*, *i*-th *child*
- [Sadakane, Navarro 10]
  - simpler and smaller

Additional Basic Operations on BP

- $rank_p(P,i)$: number of pattern p in $P[1..i]$
- $select_p(P,i)$: position of i-th occurrence of p in P
- If the length of p is constant, *rank*/*select* is done in O(1) time

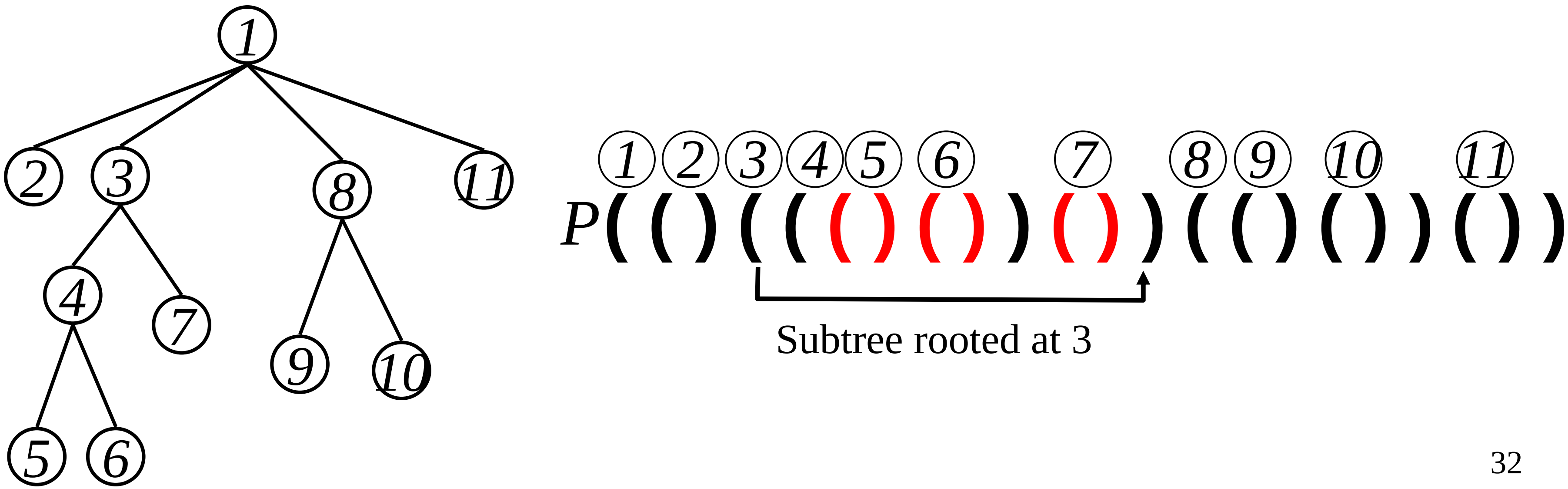


P (( ) ((( ) ( )) ( )) (( ) ( )) ( ))

$rank_{()}(P,10) = 3$

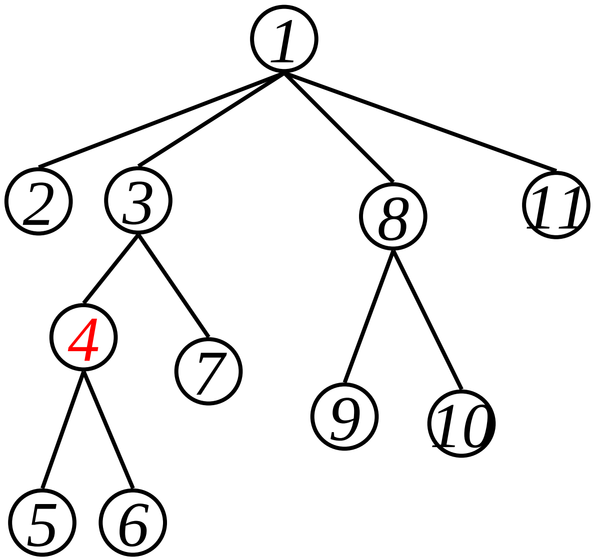
Operations on Leaves

- Each leaf is represented by **( )** in BP
- Position of *i*-th leaf = $select_{()}(P, i)$
- Number of leaves in a subtree, leftmost/rightmost leaf in a subtree are also found



Node Depths

- Define excess array $E[i] = rank_{(}(P,i) - rank_{)}(P,i)$
  $depth(v) = E[v]$
- E is not explicitly stored; it can be computed by the *rank* index on P



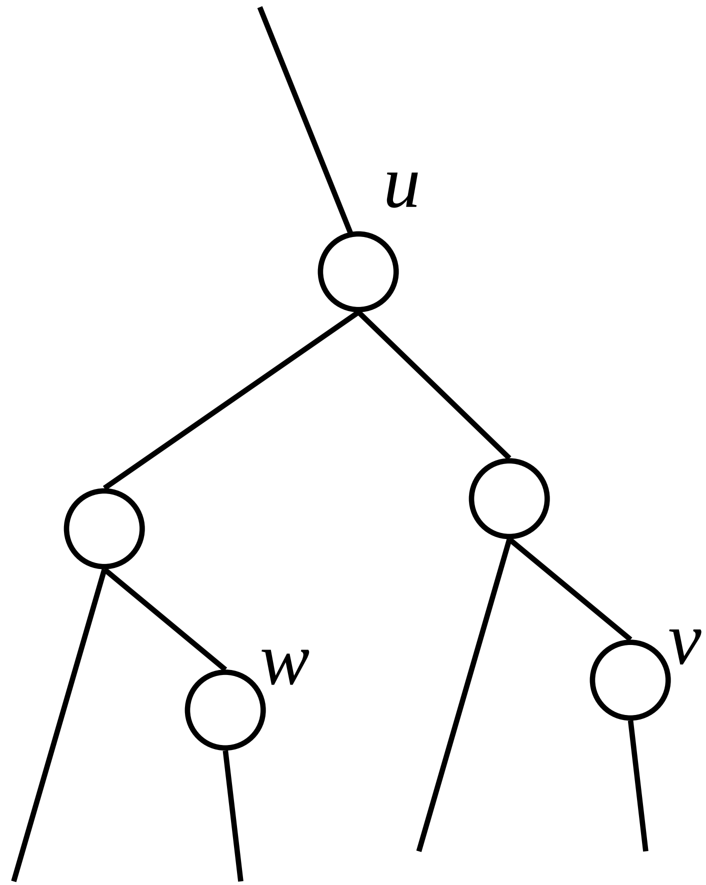
1 2 3 4 5 6 7 8 9 10 11

P `(()((()())())(()())())`

E `1212343432321232321210`

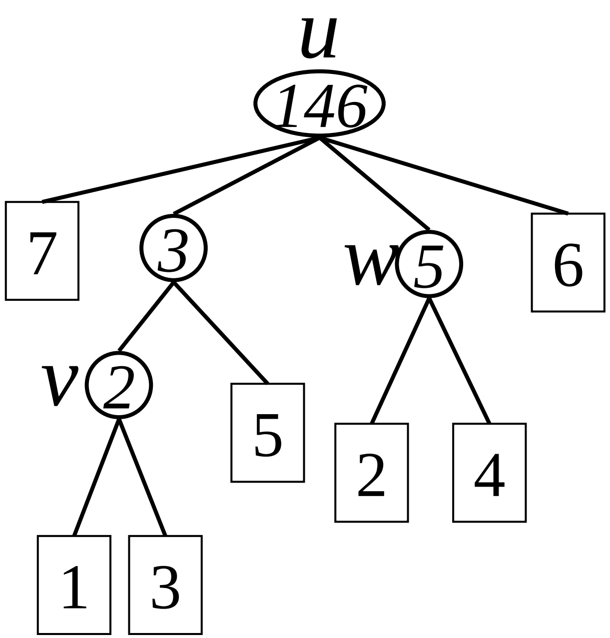
Lowest Common Ancestor (*lca*)

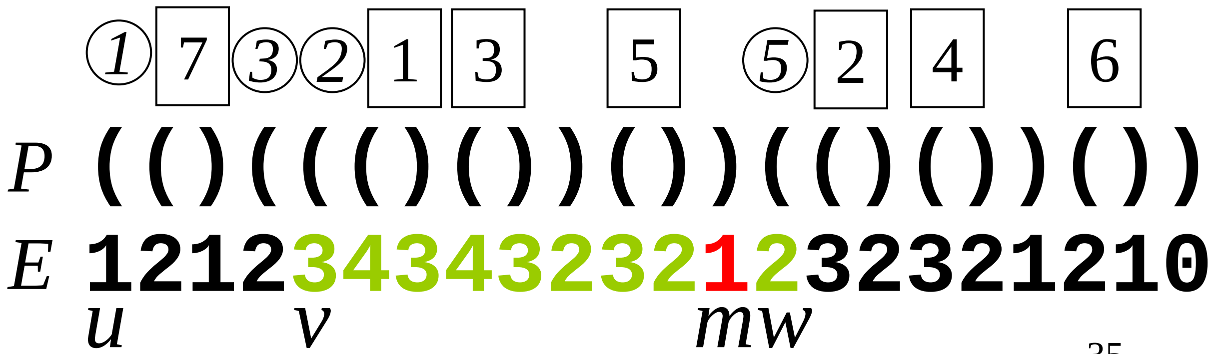
- *lca* = lowest common ancestor
- *u* = *lca*(*v*,*w*): common ancestor of *v* and *w* which is furthest from root
- Found in O(1) time



- $u = parent(RMQ_E(v,w)+1)$
  - E is the excess array, which represents node depths

$m = RMQ_E(v,w)$: the index of a minimum value in $E[v..w]$ (RMQ = Range Minimum Query)





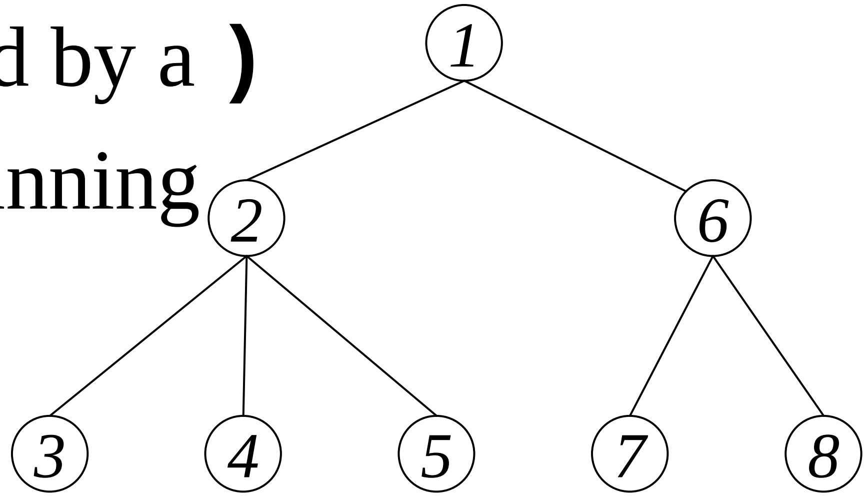
The Merits and Demerits of BP

- More supported operations
  - subtree size
  - lowest common ancestor (*lca*)
  - level ancestors

- Succinct indexes are complicated
  - the more supported operations, the more index space
  - o(*n*) size indexes cannot be ignored in practice
- *degree*, *i*-th *child*, etc. were difficult to implement
- No locality of child labels

DFUDS Representation

- It encodes the degrees of nodes in unary codes in depth-first order
  (DFUDS = Depth First Unary Degree Sequence)
- Degree $d \Rightarrow d$ **(** 's, followed by a **)**
- Add a dummy **(** at the beginning
- $2n$ bits
- DFUDS is balanced



DFUDS $\quad U \; \mathbf{(} \underbrace{\mathbf{(()}}_{\textcircled{1}} \underbrace{\mathbf{((()}}_{\textcircled{2}} \underbrace{\mathbf{)}}_{\textcircled{3}} \underbrace{\mathbf{)}}_{\textcircled{4}} \underbrace{\mathbf{)}}_{\textcircled{5}} \underbrace{\mathbf{(()}}_{\textcircled{6}} \underbrace{\mathbf{)}}_{\textcircled{7}} \underbrace{\mathbf{)}}_{\textcircled{8}}$

Various Operations on DFUDS

- A node with degree *d* is represented by the first position of its encoding **(**^{d}**)**
  - Position *v* in the sequence for a node and its preorder *i* is converted each other by

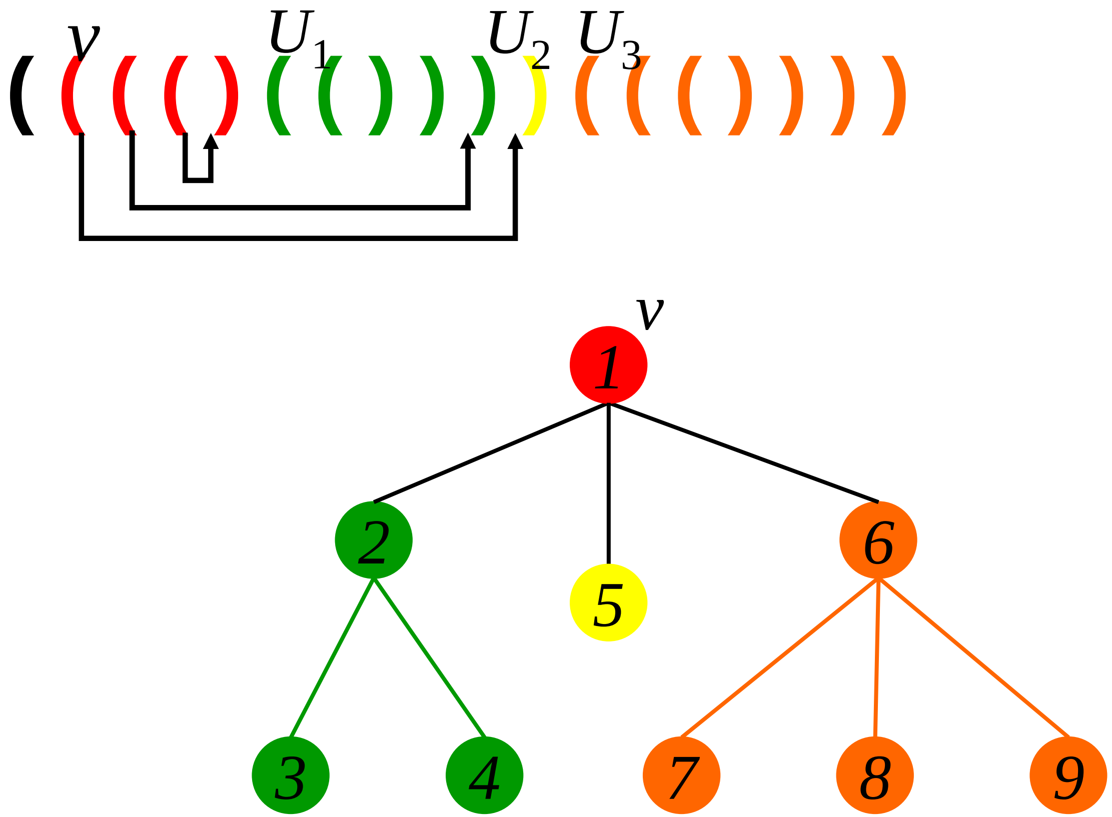
$$v = preorder\text{-}select(i) = \left(select_{)}(i-1)\right) + 1$$

$$i = preorder\text{-}rank(v) = \left(rank_{)}(v-1)\right) + 1$$

- Degree: $degree(v) = select_{)}(rank_{)}(v)+1) - v$

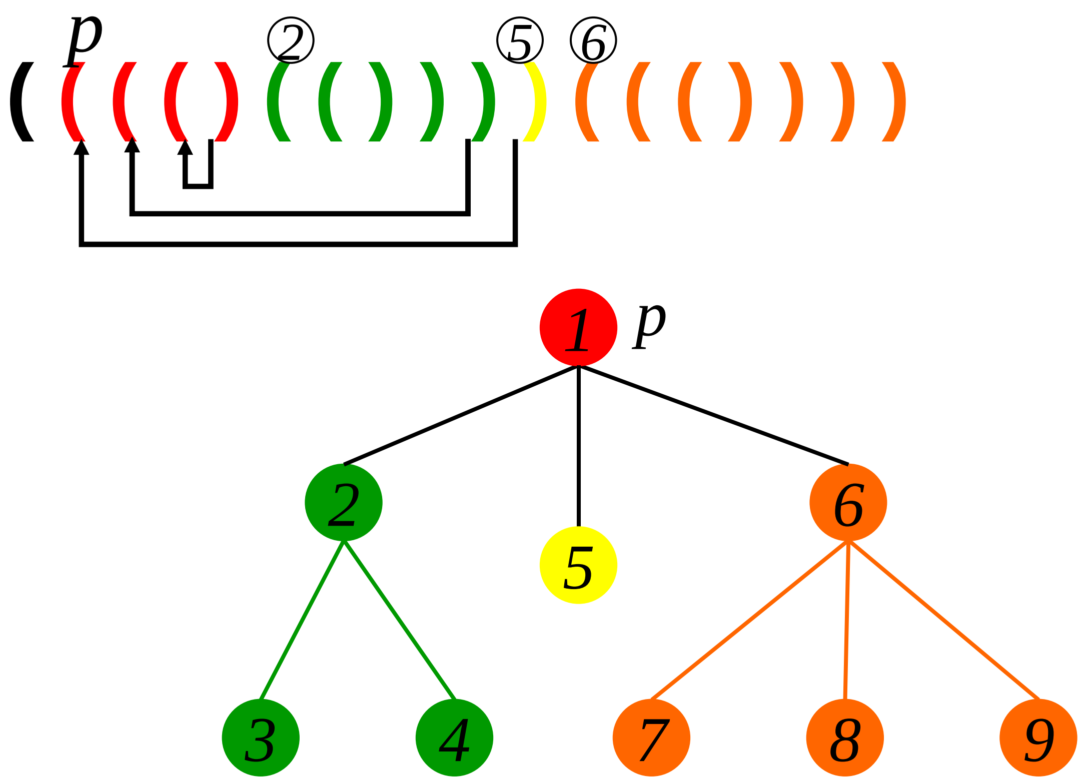
i-th child

$$child(v,i) = findclose\big(select_{)}\big(rank_{)}(v)+1\big)-i\big)+1$$



Parent

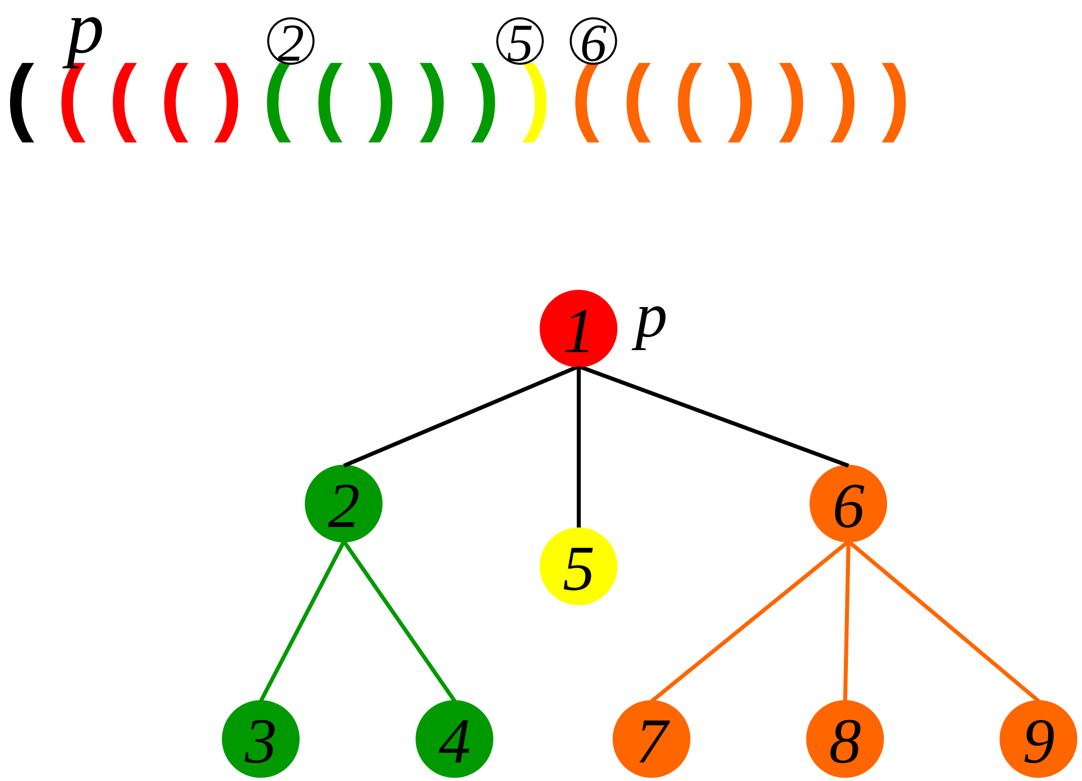
$$parent(v) = select_{)}\left(rank_{)}\left(findopen(v-1)\right)\right)+1$$



Number of Descendants (Subtree Size)

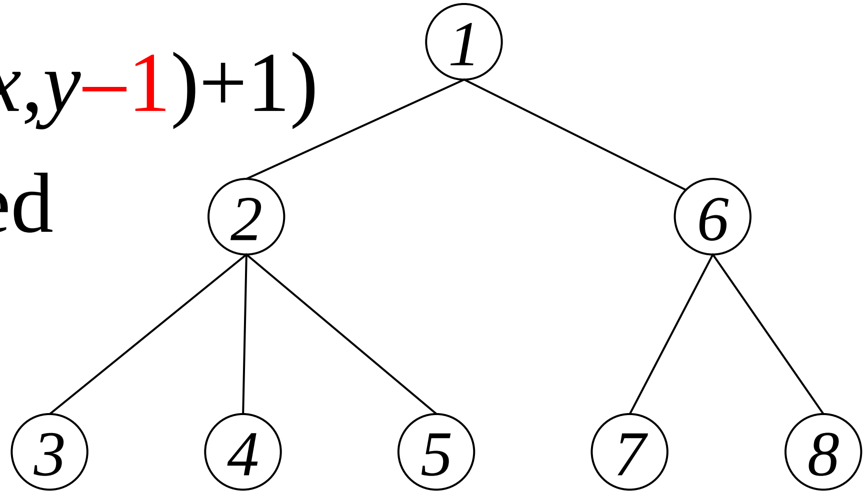
- Size of a subtree rooted at *v* is

$$subtreesize(v) = (findclose(enclose(v)) - v)/2+1$$



LCA on DFUDS

- Can be computed by almost the same operation for BP
- $lca(x,y) = parent(RMQ_E(x,y-1)+1)$
- Leftmost minimum is used



DFUDS

E 1232345432123210

U ((()((())))(()))

①②③④⑤⑥⑦⑧

BP

P ((()()())(()()))

E 1232323212323210

Tries using DFUDS

- child(v, c)

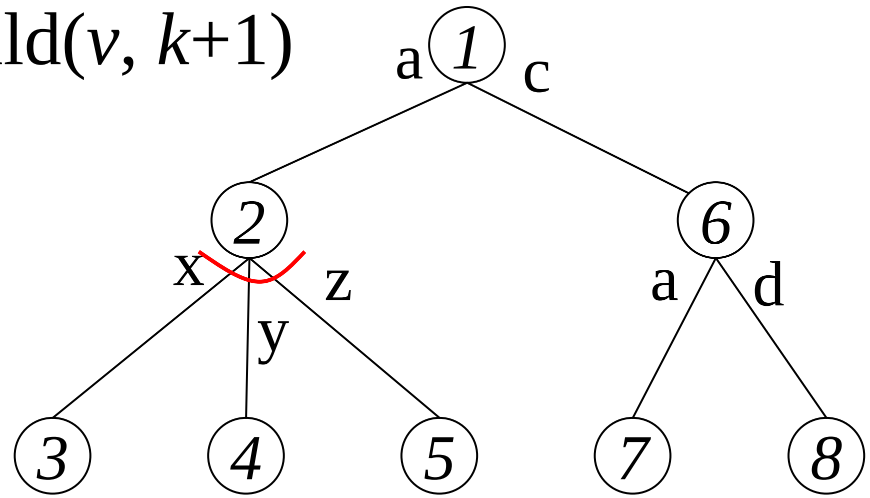
  $r = rank_{(}(U, v)$, $k = 0$

  while ($U[v+k]$!= ‘)’) {

  if ($C[r+k] == c$) return child(v, k+1)

  $k = k+1$

  }



- O(σ) time
  O(log σ) is possible

DFUDS U ((( ) ((( )))) (( )))

① ② ③④⑤ ⑥ ⑦⑧

C _ a c x y z a d

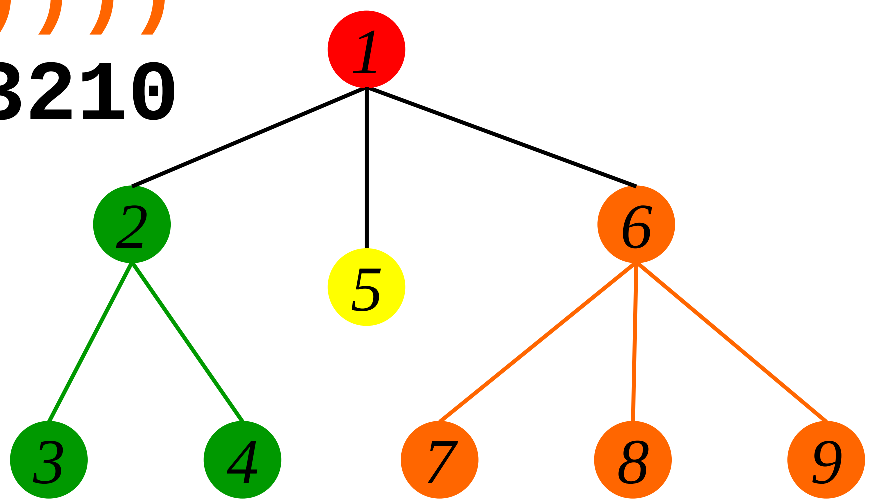
Other Operations 1

- $leaf\text{-}rank(v) = rank_{))}(v)$
- $leaf\text{-}select(i) = select_{))}(i)$
- $preorder\text{-}rank(v) = (rank_{)}(v-1))+1$
- $preorder\text{-}select(i) = (select_{)}(i-1))+1$

① ② ③④⑤ ⑥ ⑦⑧⑨

U (((( ) (( )))) ((( ))))

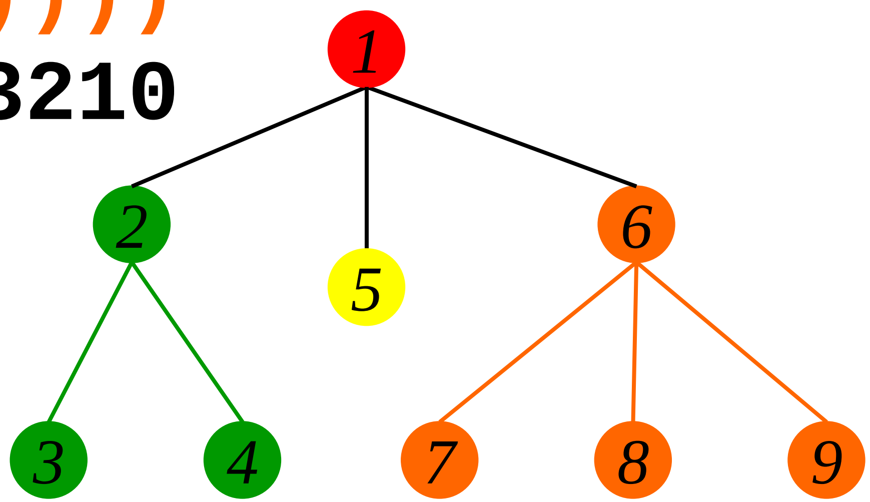
E 123434543212343210



Other Operations 2

- *inorder-rank*(*v*) = *leaf-rank*(*child*(*v*,2)–1)
- *inorder-select*(*i*) = *parent*(*leaf-select*(*i*)+1)
- *leftmost-leaf*(*v*) = *leaf-select*(*leaf-rank*(*v*–1)+1)
- *rightmost-leaf*(*v*) = *findclose*(*enclose*(*v*))

```
    ①   ②  ③④⑤ ⑥  ⑦⑧⑨
U (((()(())))((()))))
E 123434543212343210
```



The Merits and Demerits of DFUDS

- More supported operations
  - subtree size
  - lowest common ancestor (*lca*)
- Locality of child labels
- Can be compressed into less than 2*n* bits!

- Succinct indexes are complicated
  - depth
  - level-ancestor

Entropy of Sets

- # of sets $S \subseteq \{1,2,...,n\}$ with cardinality m is $\binom{n}{m}$
- We define the entropy of S as

$$nH_0(S) = m\log\frac{n}{m} + (n-m)\log\frac{n}{n-m} \approx \log\binom{n}{m}$$

- Fully Indexable Dictionary [RRR02]
  - $nH_0(S)$+O(n log log n/log n) bits
  - *member*/*rank*/*select*: constant time

Entropy of Ordered Trees

- Number of ordered trees having n_i nodes with degree i (i=0,1,...,n) is

$$L = \frac{1}{n}\binom{n}{n_0 \quad n_1 \quad \cdots \quad n_{n-1}}$$ [Rote 96]

- The ITLB of trees having the degree sequence is

$$\log L \approx \sum_i n_i \log \frac{n}{n_i}$$

- We define *tree degree entropy* as $H^*(T) = \sum_i \frac{n_i}{n} \log \frac{n}{n_i}$
- $nH^*(T) \leq 2n + o(n)$ for any tree

Trees with Low Entropy

- A binary ordered tree

$$\sum_{i=0}^{2} n_i \log \frac{n}{n_i} \le n \log 3 \approx 1.58n < 2n$$

- Full binary tree (A tree in which all internal nodes have exactly two children)

$$\underbrace{\frac{n-1}{2} \log \frac{2n}{n-1}}_{\substack{\text{internal nodes} \\ \text{(degree 2)}}} + \underbrace{\frac{n+1}{2} \log \frac{2n}{n+1}}_{\substack{\text{leaves} \\ \text{(degree 0)}}} \approx n < 2n$$

How to Achieve the Entropy

- The tree degree entropy is identical to the order-0 entropy of the degree sequence *S* of the tree

$$H_0(S) = H^*(T) = \sum_i \frac{n_i}{n} \log \frac{n}{n_i}$$

- Any log *n* bits of *U* are recovered from *S* in O(1) time. → *U* can be regarded as uncompressed.

DFUDS U $(\underbrace{(()}_{①}\underbrace{((()}_{②}\underbrace{)}_{③}\underbrace{)}_{④}\underbrace{)}_{⑤}\underbrace{(()}_{⑥}\underbrace{)}_{⑦}\underbrace{)}_{⑧}$

New S **2 3 0 0 0 2 0 0**

Simpler Indexes

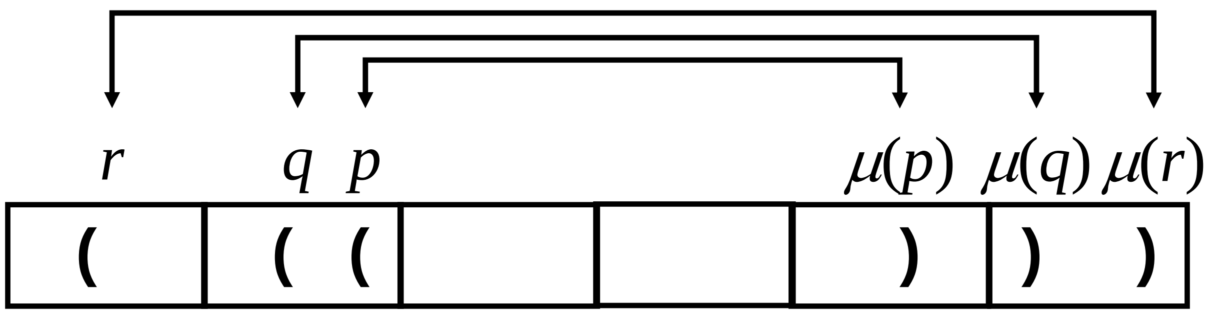
- [Geary, Rahman, R. Raman, V. Raman 04, 06]
  - recursive representation
- [Sadakane, Navarro 10]
  - range min-max tree

Data Structure for *findclose*

- Divide the parentheses sequence into blocks of length $B = ½ \log n$
  - $b(p)$: block number containing p
  - $\mu(p)$: position of parenthesis matching p
  - parenthesis p is said to be far $\Leftrightarrow b(p) \neq b(\mu(p))$
- Far open parenthesis p is said to be opening pioneer

$\Leftrightarrow$ For the far open parenthesis q which is immediately precedes p, $b(\mu(p)) \neq b(\mu(q))$

- Represent positions of parentheses which match with opening pioneers are represented by 0,1 vector



Lemma: Let β denote the number of blocks. Then the number of opening pioneers is at most $2\beta - 3$.

Proof: A graph whose nodes correspond to the blocks and whose edges are $(b(p), b(\mu(p))$ is an outer-planar graph.

Opening/closing pioneers form a BP again.

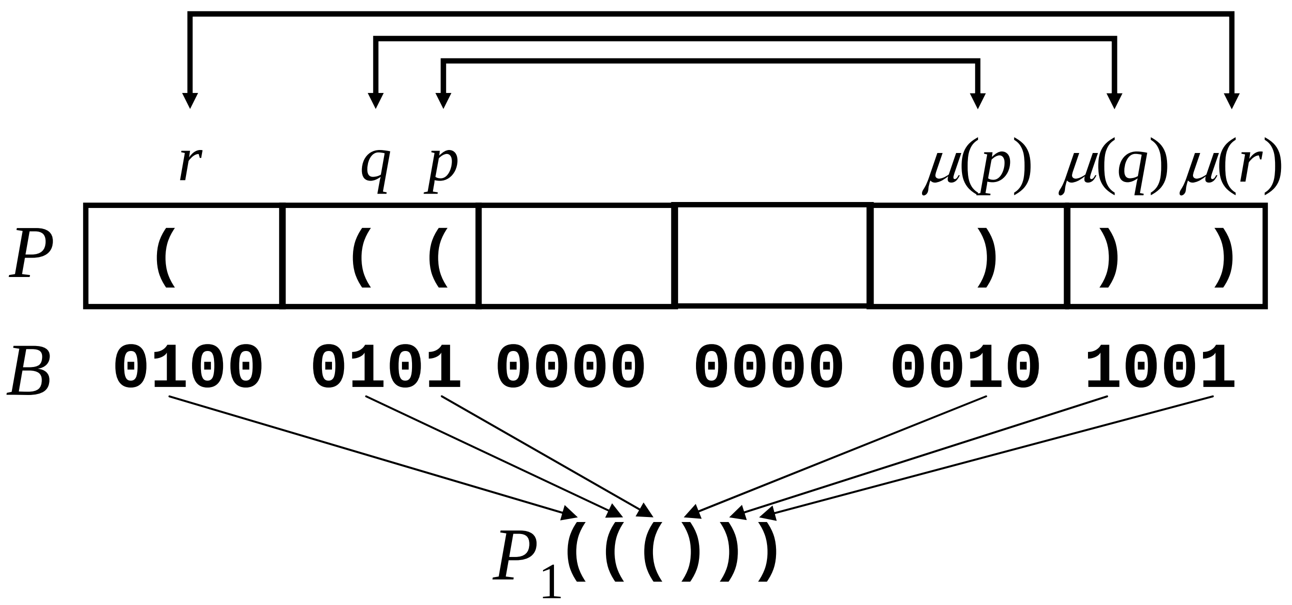
$\beta = n/B = 2n/\log n \Rightarrow$ Length of BP is $O(n/\log n)$

Representing Recursive Structure

- opening pioneers and their matching parentheses are represented by a 0,1 vector B

$$\mu(p) = select\left(B, findclose\left(P_1, rank\left(B, p\right)\right)\right)$$

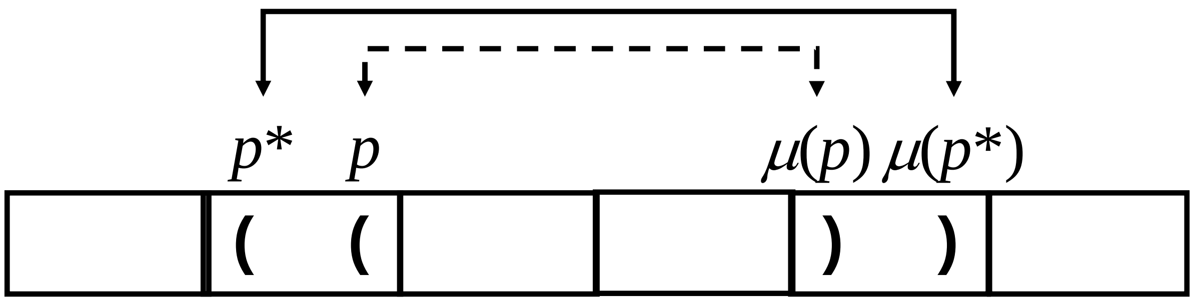
- B is a sparse vector of length $2n$ with $O(n/\log n)$ 1's
  - Can be represented in $O(n \log\log n/\log n)$ bits



- Let $S(n)$ denote the size of BP representation for an n node *tree*
  - $S(n) = 2n + O(n \log \log n / \log n) + S(O(n/\log n))$
- If the number of nodes becomes $O(n/\log^2 n)$, a naïve data structure which stores all the answers uses only $O(n/\log n)$ bits
- Therefore $S(n) = 2n + O(n \log \log n/\log n)$

Algorithm for *findclose*

- To compute $\mu(p) = findclose(P,p)$
- If p is not far, $\mu(p)$ is computed by a table
- Find the pioneer p^* that immediately precedes p
- Find $\mu(p^*)$ using the BP for pioneers
- If p is not pioneer, $b(\mu(p)) = b(\mu(p^*))$
- The position of $\mu(p)$ is determined from the difference between depths of p and p^*



Range Min-Max Trees

- Existing approach uses a distinct data structure for each operation.
- The more functions, the larger indexes.

- New approach uses basically only one basic data structure (range min-max tree) which supports various operations.

How to implement tree operations

- Def

$$fwd\_excess(E,i,d) \stackrel{\text{def}}{=} \min_{j>i}\{j \mid E[j]=E[i]+d\}$$

$$bwd\_excess(E,i,d) \stackrel{\text{def}}{=} \max_{j<i}\{j \mid E[j]=E[i]+d\}$$

- Lemma

$$findclose(P,i) = fwd\_excess(E,i,-1)$$

$$findopen(P,i) = bwd\_excess(E,i,0)+1$$

$$enclose(P,i) = bwd\_excess(E,i,-2)+1$$

$$level\_ancestor(P,i,d) = bwd\_excess(E,i,d-1)+1$$

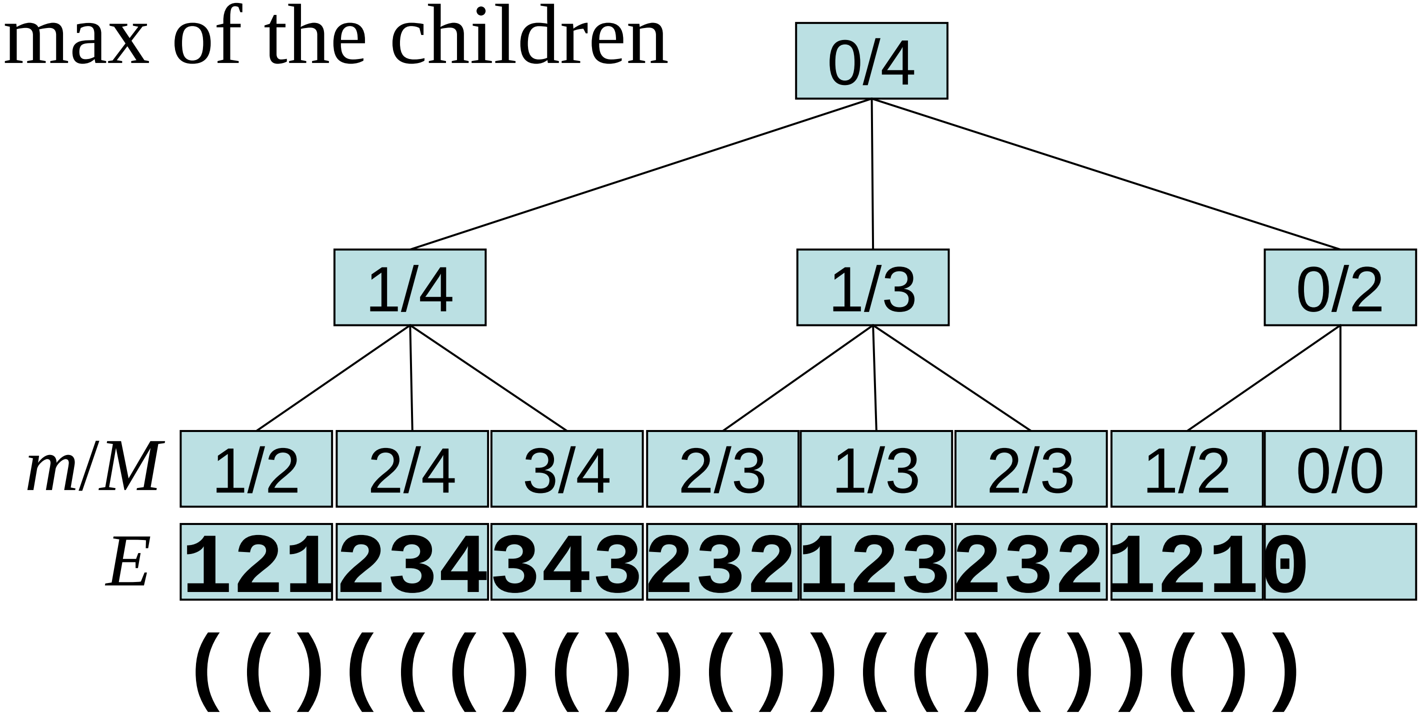
enclose findclose

P `(()((()()))(()()))(()))`

E **121234343232123232121 0**

Range min-max tree

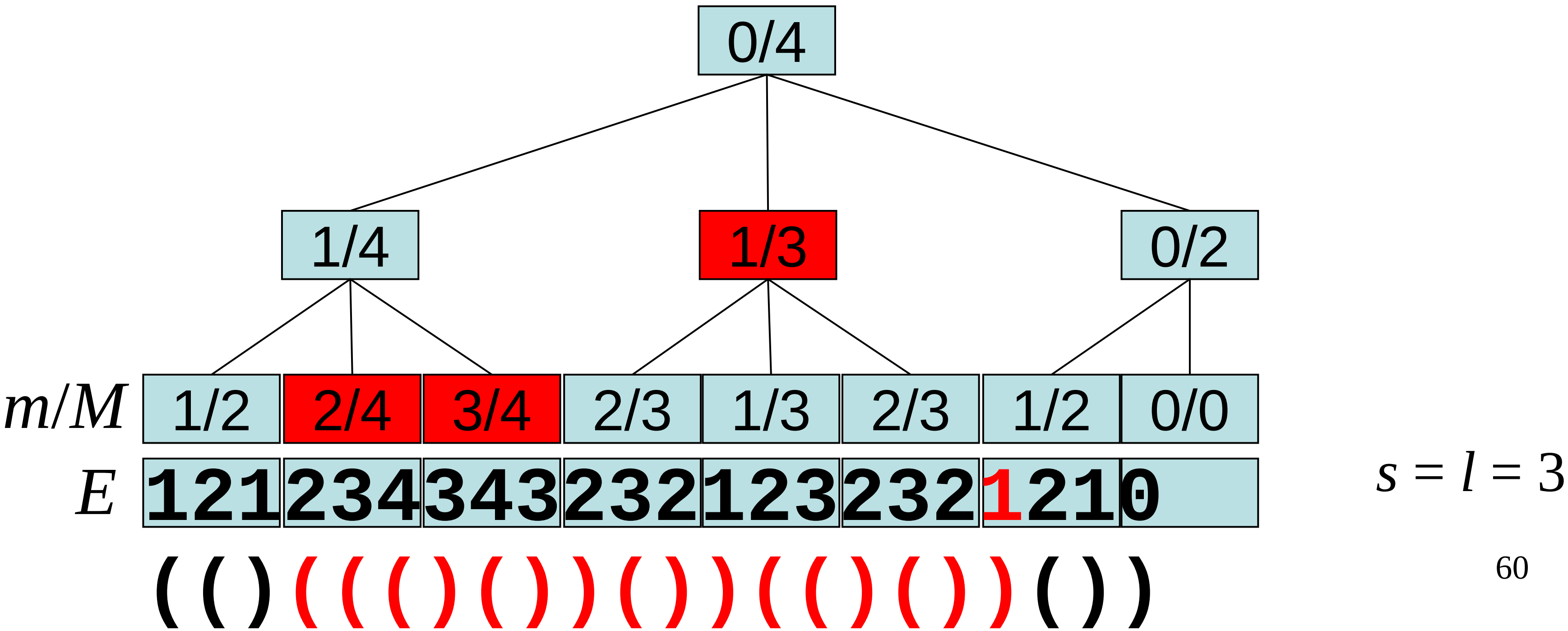
- Partition excess array E into blocks of length s
- Each leaf corresponds to a block, and stores min and max in the block
- Each internal node has l children and stores min and max of the children



$s = l = 3$

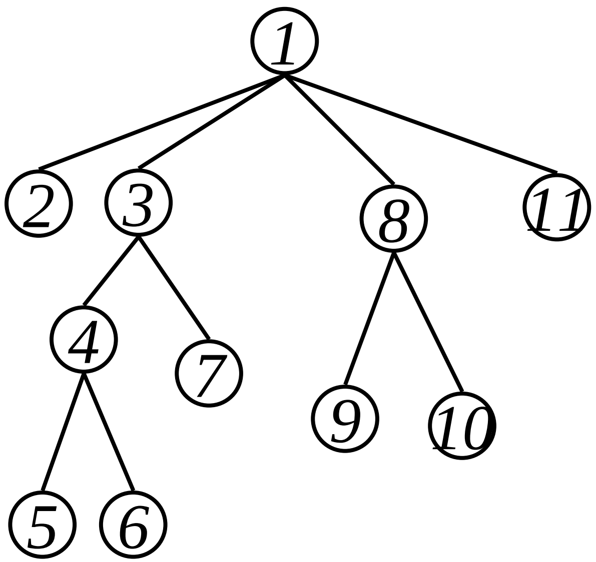
Properties of range min-max tree

- Each node corresponds to a range of excess array
- Any interval of excess array is decomposed into O(lh) intervals corresponding to internal nodes, and at most two sub-intervals of leaves (h: tree height)



Properties of excess array

- For any i, either $E[i+1] = E[i]-1$ or $E[i]+1$
- Let a, b be min and max of interval $E[u,v]$. Then in the interval all integers $a \le e \le b$ exists.
- Difference between min and max in an interval of length l is at most $l-1$



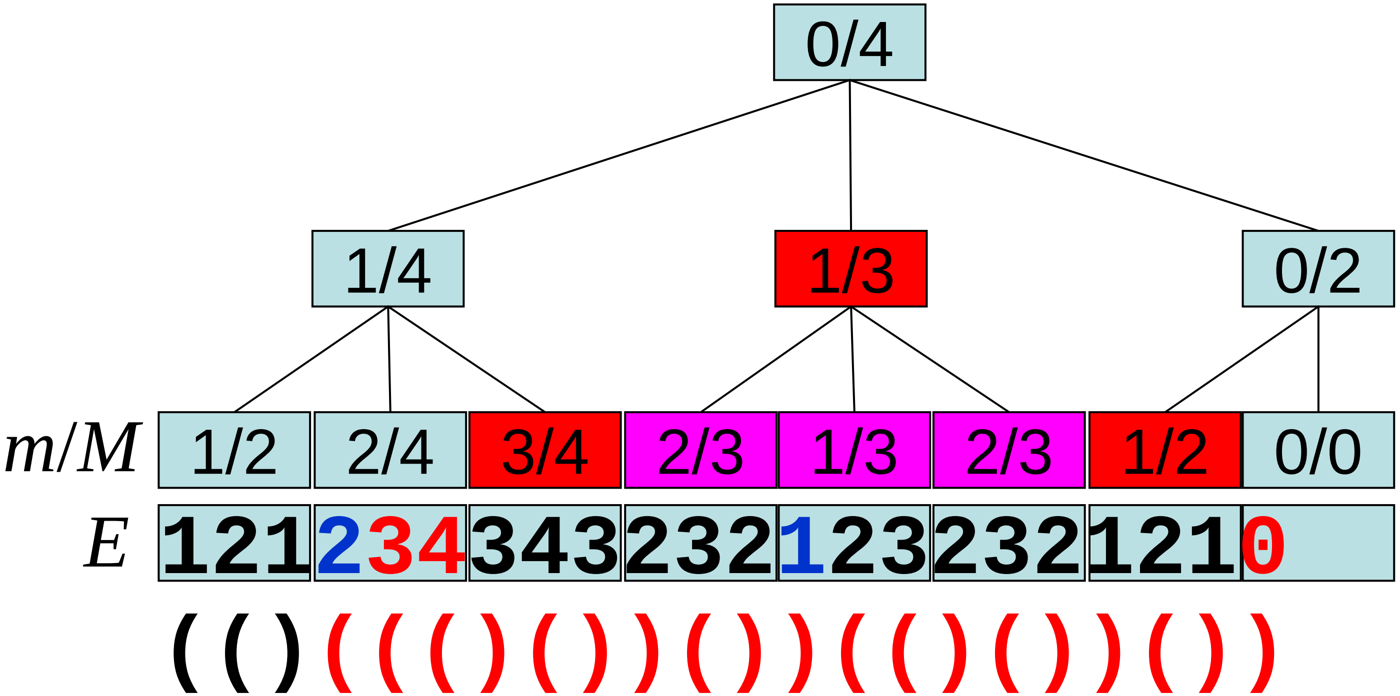
1 2 3 4 5 6 7 8 9 10 11

P `(()((()())())(()())())`

E `1212343432321232321210`

How to compute *fwd_excess*(E,i,d)

- Decompose $E[i+1,N-1]$ into sub-intervals
- Scan the intervals from left to right, and find the interval containing $E[i]+d$
- O($lh+s$) time

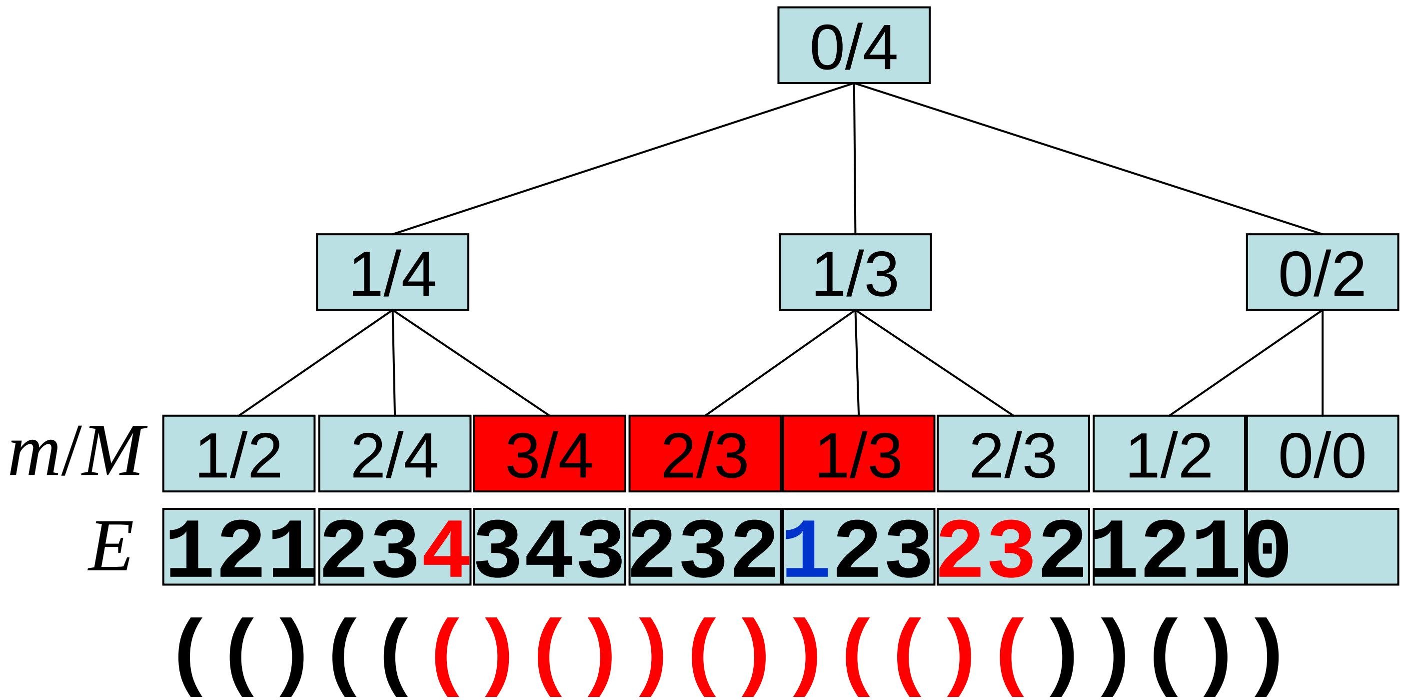


The case sequence is short

- Let w be the length of machine word
- Lemma
  If $N < w^c$, *fwd_excess* is computed in $O(c^2)$ time
  Size of data structure is $N + O(Nc/w) + \exp(w)$ bits
- Proof: $l = \Theta(w/\log w)$, $s = \Theta(w)$
  tree height $h = O(c)$. Traverse the tree from root.
  To find the child containing the target,
  examine l/c children at a time.
  min/max value can be stored in $c \log w$ bits
  $\Rightarrow \Theta(w)$ bits for l/c children
  The target can be found in $O(1)$ time.

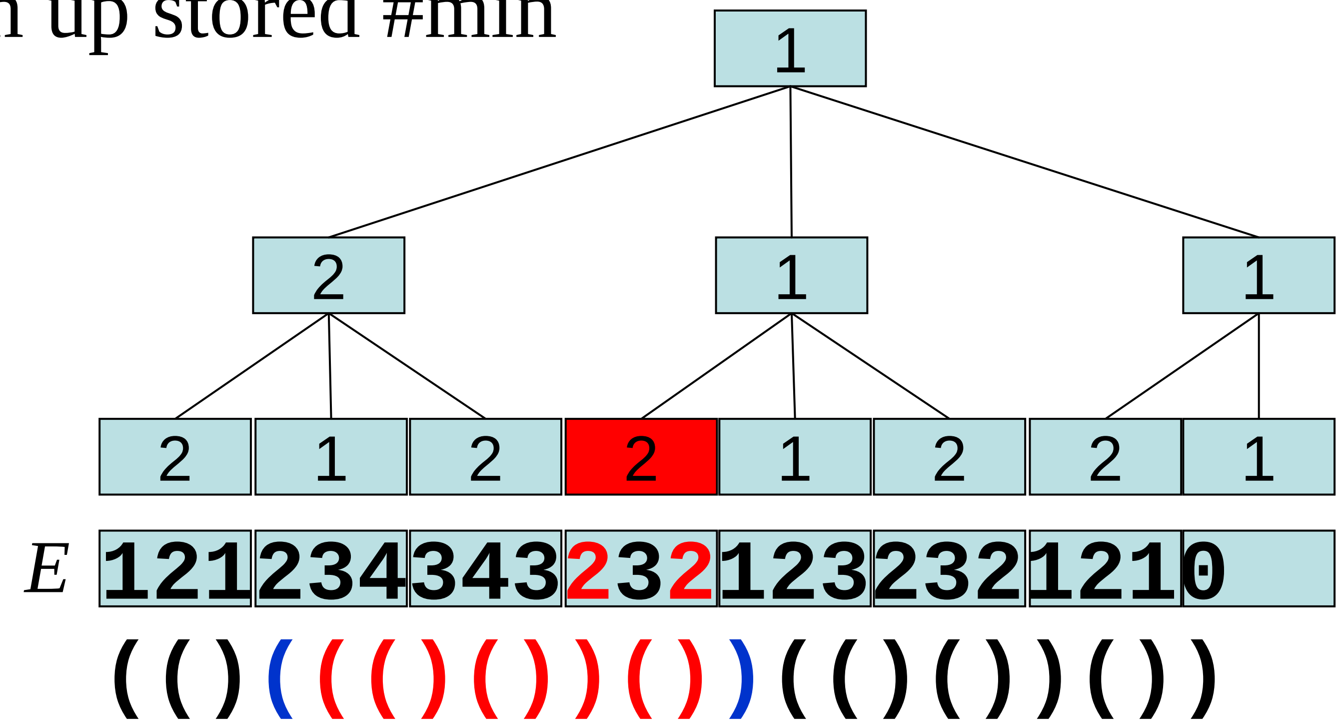
How to find lowest common ancestor

- *lca*(*v*,*w*) = *parent*(*RMQ*(*v*,*w*)+1)
  - *RMQ*: position of min in *E*[*v*,*w*]
- O(1) time using range min-max tree



How to compute degree

- Let $E[v,w]$ be the interval for node v
- $deg(v)$ = (number of min in $E[v+1,w-1]$)
- For each node of range min-max tree, store the number of min in the corresponding interval
- Sum up stored #min



For long sequences

- Contract tree, and apply existing data structures
  - remove subtrees with < w^c nodes
  - size of auxiliary data structures can be ignored
- For parent operation, pioneers are used
- For lca, the sparse table algorithm is used

Sparse Table Algorithm

- For each interval $[i,i+2^k-1]$ in array $B[1,m]$, store the minimum value in $M[i,k]$.

  ($i = 1,\ldots,m$, $k = 1,2,\cdots,\lg m$)

- For a given query interval $[s,b]$
  1. Let $k = \lg(b-s)$
  2. Compare $M[s,k]$ and $M[b-2^k+1,k]$, and output the minimum.

- O(1) time, $O(m \lg^2 m)$ bit space

0 3

B | 143504537 |

- This data structure is used when the length of B becomes $O(n/\lg^3 n)$

  $\Rightarrow o(n)$ bit space

- Main theorem: There exists a data structure for an n-node ordinal tree supporting all operations in O(1) time using $2n + O(n/\log^c n)$ bits for any $c > 0$.

- In practice, the range min-max tree for large blocks is enough. This leads to an extremely simple implementation.

Operations supported in basic structure (bpmin) . Size 2.30n bits

Operation	Description
findclose/findopen	position of matching parenthesis
enclose	position of minimum enclosing parentheses
preorder rank/postorder rank	preorder/postorder of a node
inspect(i)	returns P [i]
isleaf (i)	returns yes if P [i] is leaf
isancestor(x, y)	returns yes if x is an ancestor of y
depth	depth of node
parent	parent node
first child/last child	first/last child of a node
next sibling/prev sibling	younger/older brother
subtree size	number of nodes in a subtree
level ancestor(u, d)	node v s.t. v is an ancestor of u and depth(v)= depth(u)-d
level next/level prev	next/prev node in BFS order
level leftmost/level rightmost	leftmost/rightmost node in given depth
lca(x, y)	lowest common ancestor of nodes x and y
deepest node	node with maximum depth in a subtree
preorder select/postorder select	node with given preorder/postorder (slow)
degree	number of chidren (slow)
child(u, i)	i-th child of node u (slow)
child rank	number of older brothers (slow)

Operations supported in extended data structure (bpmax).		
Operation	Description	Size
degree	number of children (fast)	0.08*n*
child	i-th child (fast)	
child rank	number of older brothers (fast)	
preorder/postorder select	node with given preorder/postorder (fast)	
inorder rank	inorder of node	0.08*n*
inorder select	node with given inorder (slow)	
inorder select	node with given inorder (fast)	0.04*n*
leaf rank	number of leaves with preorder < i	0.08*n*
leaf select	i-th leaf (slow)	
leftmost leaf	leftmost leaf in a subtree (slow)	
rightmost leaf	rightmost leaf in a subtree (slow)	
leaf select	i-th leaf (fast)	0.04*n*
leftmost leaf	leftmost leaf in a subtree (fast)	
rightmost leaf	rightmost leaf in a subtree (fast)	

Practical Implementations

- [Gonzalez, Grabowski, Makinen, Navarro 05]
  - *rank, select*
- [Kim, Na, Kim, Park 05]
  - *rank, select*
- [Delpratt, Rahman, R. Raman 06]
  - LOUDS++
- [Okanohara, Sadakane 07]
  - *rank, select* on compressed vectors
- [Valimaki, Makinen, Gerlach, Dixit 09]
  - compressed suffix trees
- [Navarro 09]
  - Hash based BP
- [Arroyuelo, Canovas, Navarro, Sadakane 10]
  - tries
- [Gog, Ohlebusch 11]
  - compressed suffix trees
- [Brisaboa, Canovas, Claude, Martinez-Prieto, Navarro 11]
  - compressed string dictionaries
- [Grossi, Ottaviano 12]
  - compressed tries through path decomposition

Empirical Comparison

- Size for *n*-node ordered trees
  - LOUDS: 2.10*n* bits
  - BP, DFUDS: 2.37*n* bits
- BP is faster than others for all operations except child(*v*, *c*) and *i*-th child
- For child(*v*, *c*) and *i*-th child, LOUDS is the fastest, DFUDS is the next, because of locality of access to the label array.

Comparison with Non-Succinct Trees

- Non-succinct trees are 12-20 times faster than LOUDS for *i*-th child.
- Non-succinct trees supporting only *i*-th child are 13 times larger than BP, 15 times larger than LOUDS.
- Trees supporting parent, depth, subtreesize, preorder use 66 times larger space than BP.
- Supporting lca, level_ancestor uses more space…

- Succinct trees support a wide set of functions using small space.

Heavy Path Decomposition

- A rooted tree is decomposed into heavy paths so that any leaf-to-root path consists of at most O(log n) heavy paths.
- Each heavy path is stored consecutively.
  - Good locality of reference
  - BP based on Range Min-Max Tree is used.
- Fast string dictionaries

Concluding Remarks

- Succinct trees are practical now.
  - BP supports various operations using small space.
  - LOUDS is fast for child operations.
  - Heavy path decomposition is effective for string dictionaries.
- Future work: practical implementations for dynamic labeled trees.
  - Range Min-Max Tree is easy to dynamize.