Induced immersions

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Minato ERATO Workshop, Rebun island, September 9th, 2014

Some graph operations

- Vertex deletion;
- Edge deletion;
- Edge contraction;
- Vertex dissolution;
- Lift.

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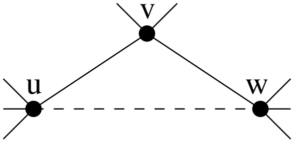

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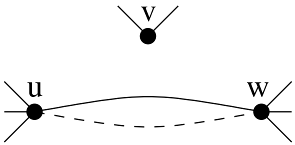

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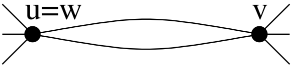

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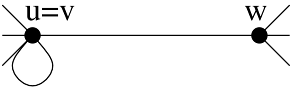

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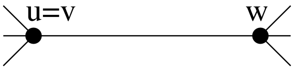

Containment relations

Containment Relation	VD	ED	EC	VDi	L
Minor	yes	yes	yes	yes	no
Induced minor	yes	no	yes	yes	no
Topological minor	yes	yes	no	yes	no
Induced topological minor	yes	no	no	yes	no
Immersion	yes	yes	no	yes	yes
Induced immersion	yes	no	no	yes	yes

Table : Some containment relations.

Some terminology

Definition

A containment relation $R(G, H)$ is:

- FPT if there exists an algorithm that decides R in time $f(|H|) \cdot poly(|G|)$, for some function f and every pair of graphs G and H;
- XP if there exists an algorithm that decides R in time $|G|^{f(|H|)}$, for some function f and every pair of graphs G and H;
- NP-complete for fixed H (aka paraNP-complete) if there exists a fixed graph H for with deciding $R(G, H)$ is NP-complete;

Complexity of containment relations

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Immersion	yes	yes	no	yes	yes	FPT
Induced immersion	yes	no	no	yes	yes	?

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Theorem (B., van 't Hof and Kamiński, ISAAC'12)

*For every simple graph H, the H-*INDUCED IMMERSION *problem can be solved in polynomial time.*

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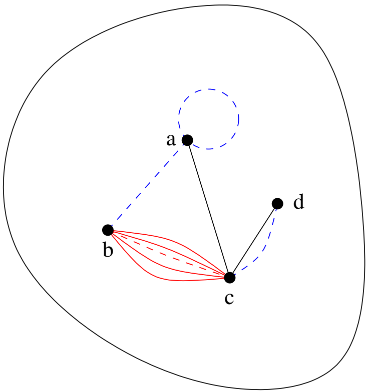
Our result

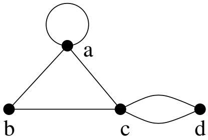
Theorem

For every **multigraph** H*, the* H-INDUCED IMMERSION *problem can be solved in polynomial time.*

An example



G



H

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Question: How do we bound $|E(G[S])|$?

Alternative definition for immersion

Observation

Let G and H be two multigraphs. G contains H as an immersion if and only if there exists a set of vertices S of G, a bijection ϕ from $V(H)$ to $V(G)$, and a map α from $E(H)$ to paths in G such that:

- *for every edge e of H with endpoint u, v, the path $\alpha(e)$ has endpoints $\alpha(u), \alpha(v)$;*
- *for every $e \neq f \in E(H), \alpha(e)$ and $\alpha(f)$ are edge-disjoint.*

Alternative definition of induced immersion

Theorem

Let G and H be two multigraphs. Then G has an H-model $(S, \mathcal{L}, \phi)$ if and only if there exists a collection $\mathcal{A} = \{A_e \mid e \in E(H)\}$ of non-empty trails satisfying the following:

(i) *for every $e \in E(H)$ with endpoints u and v, A_e has endpoints $\phi(u)$ and $\phi(v)$;*

(ii) *for every $e, e' \in E(H)$ with $e \neq e'$, A_e and $A_{e'}$ are edge-disjoint;*

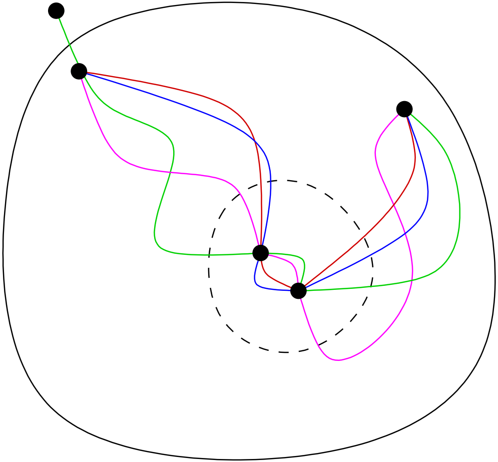
(iii) *the edges in $E(G[S]) \setminus \bigcup_{e \in E(H)} E(A_e)$ can be covered by a collection $\mathcal{B}$ of mutually edge-disjoint trails in $G - \bigcup_{e \in E(H)} E(A_e)$ such that each trail in $\mathcal{B}$ has at least one endpoint in $V(G) \setminus S$.*

Bounding the number of edges in $G[S]$

Lemma

Let G and H be two multigraphs, let S be a subset of $V(G)$ of size $|V(H)|$, and let $\phi : V(H) \to S$ be a bijection. If there exists a pair $u, v \in V(H)$ such that $\text{mult}_G(\phi(u)\phi(v)) \geq 6|V(H)|^2 + 2|E(H)| + 1$, then G has an H-model $(S, \mathcal{L}, \phi)$ if and only if $G - \{e, e'\}$ has an H-model $(S, \mathcal{L}', \phi)$, where e and e' are two distinct edges between $\phi(u)$ and $\phi(v)$.

Sketch of proof



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Theorem (B., van 't Hof, Kamiński, ISAAC'12)

Let H be a multigraph with maximum degree at most 2. Every multigraph with "big" treewidth contains H as an induced immersion.

Example of similar results for other containment relations

Theorem (Robertson, Seymour, GM V)

Let $\mathcal{C}$ be a set of graphs. There exists a constant t such that every graph in $\mathcal{C}$ has treewidth at most t if and only if there exists a planar graph H such that no graph in $\mathcal{C}$ contains H as a minor.

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Let $\mathcal{C}$ be a set of graphs. There exists a constant t such that every graph in $\mathcal{C}$ has treewidth at most t if and only if there exists a subcubic planar graph H such that no graph in $\mathcal{C}$ contains H as an immersion.

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Question: What about the "if" direction?

Our result for induced immersion

Theorem

Let H be a planar multigraph with maximum degree at most 3. Every multigraph with "big" treewidth contains H as an induced immersion.

Tools for the proof: (1) The wall

Theorem (Robertson, Seymour, Thomas, Quicly Excluding a Planar Graph)

For $r \geq 1$, every graph with "big" treewidth contains W_r as an immersion.

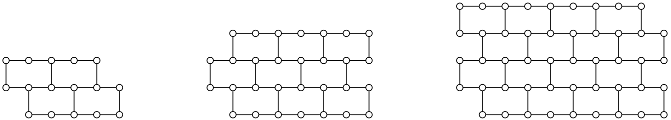

Figure : Elementary walls of height 2, 3 and 4.

Tools for the proof: Finding a large independent set

Theorem (DeVos, Dvorak, Fox, McDonald, Mohar, Scheide, Arxiv:1101.2630)

For every positive integer t, every simple graph of minimum degree at least $200t$ contains an immersion of the complete graph K_t.

Corollary

Let G be a graph such that K_t is not an immersion of G. Then G has an independent set of size $O(|V(G)|)$.

Lemma (BHK, ISAAC'12)

Let G and H be two multigraphs. If G contains a large clique as an immersion, then G contains H as an induced immersion.

Finding a large wall as an induced immersion

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Follows from the fact that every planar subcubic multigraph is an immersion of a sufficiently large wall.

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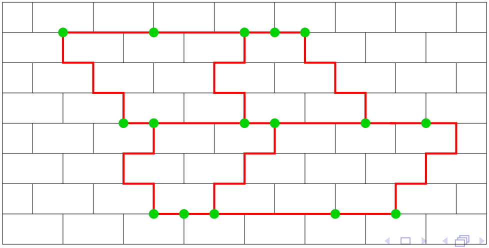

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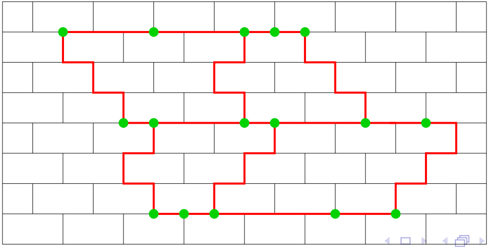
- If W contains a big clique as an immersion, then G contains H as an induced immersion. Otherwise, W must contain an independent set of size $O(|V(W)|)$;
- Since W has an independent of size $O(|V(W)|)$, W has a large "nice" independent set;
- Hence W contains a large elementary as an induced immersion, and hence it also contains H.



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- Complexity of deciding the relation;
- Structure of graphs excluding a fixed pattern;
- Well-quasi-orders.

Comparison between immersion and induced immersion

	Immersion	Induced immersion
Complexity	FPT	XP
Structure	H is planar subcubic $\Updownarrow$ bounded treewidth	H is planar subcubic $\Updownarrow$ bounded treewidth
Well-quasi-order	general graphs are WQO	?

Table : Immersion vs. induced immersion.

どうもありがとうございました

Thank you