Parameterized Edge-Hamiltonicity

Valia Mitsou

JST ERATO
Minato Project

Joint with

Michael
Lampis

Université
Paris Dauphine

Kazuhisa
Makino

RIMS, Kyoto
University

Yushi Uno

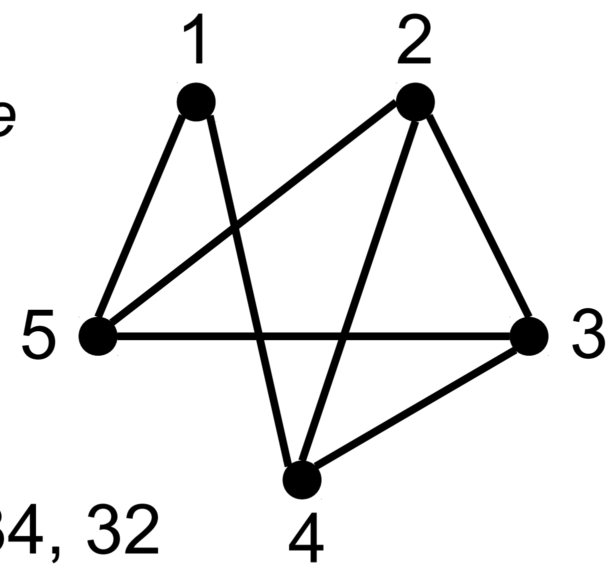
Osaka
Prefecture
University

Edge-Hamiltonian Path

EHP - Definition:

"Ordering of the edges such that consecutive edges share a common vertex."

- EHP = Hamiltonian Path on the line graph.
- EHP is different than Eulerian Path.



Edge-Hamiltonian Path: 14, 42, 25, 51, 53, 34, 32

EHP – related work

- NP-Complete even on bipartite graphs (Lai and Wei 1993) or on graphs with maximum degree 3 (Ryjacek et al. 2011).
- Demaine et. al. 2014: EHP on bipartite graphs with one constant-size part is in P.

Motivation: UNO game.

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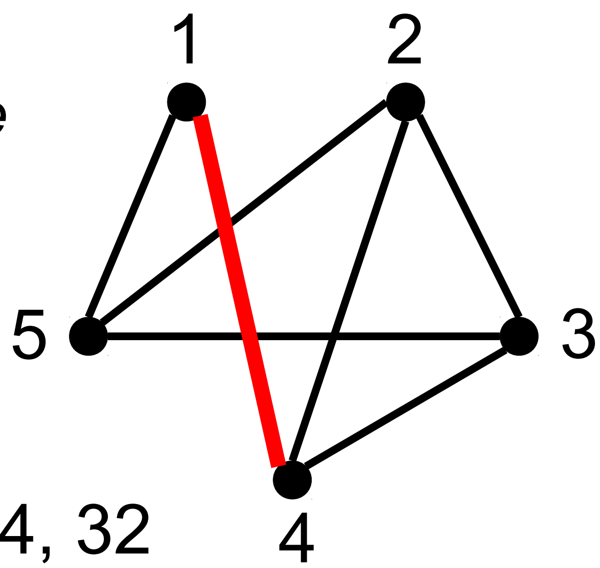

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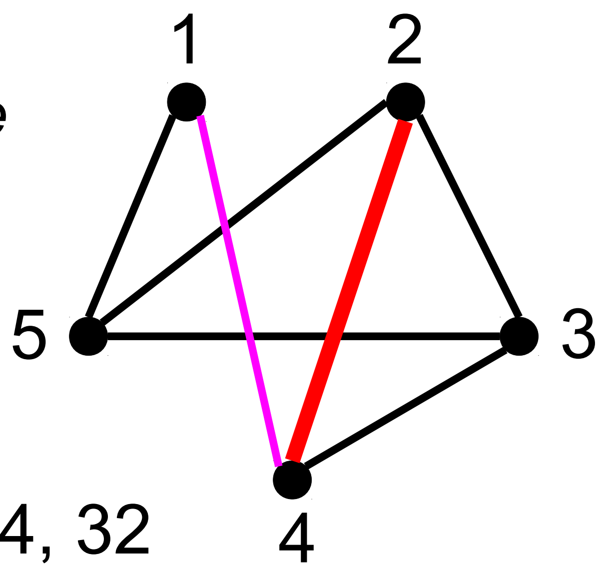

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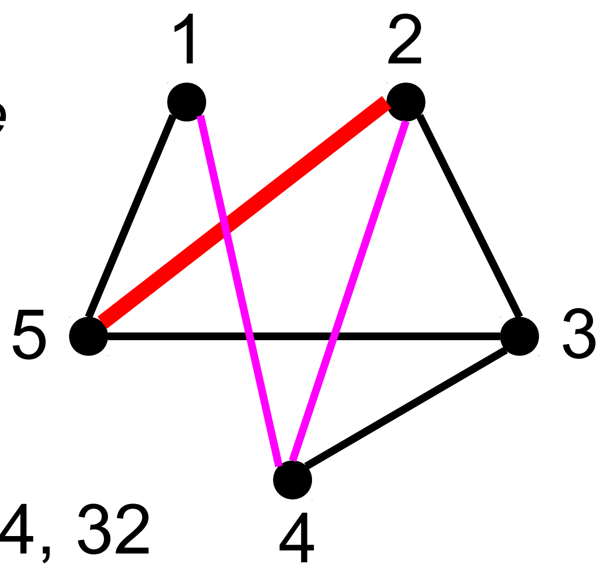

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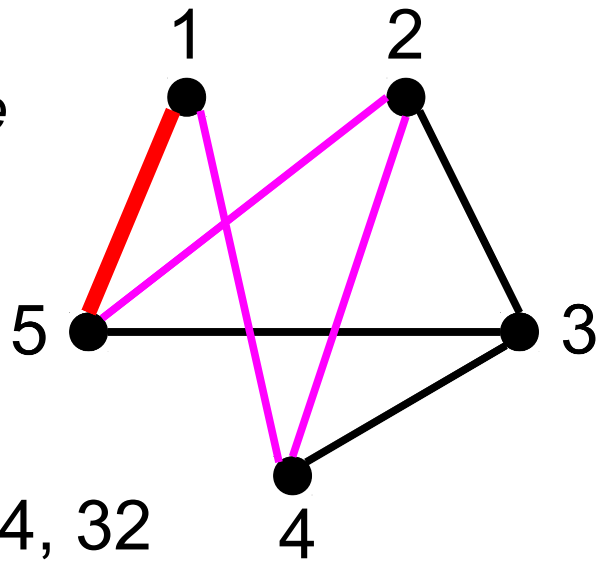

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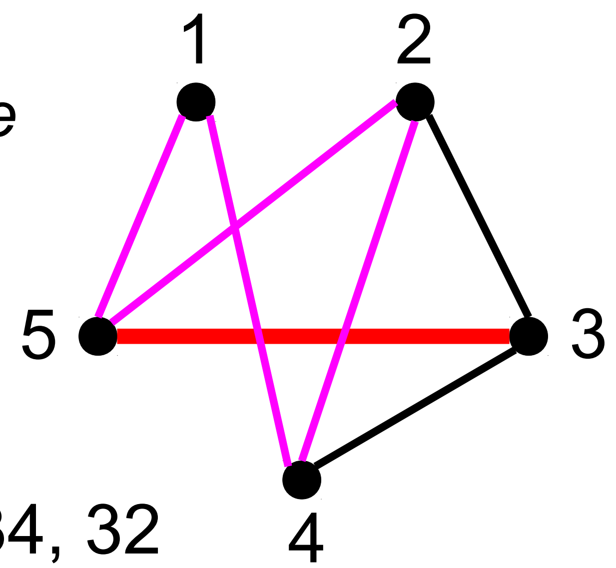

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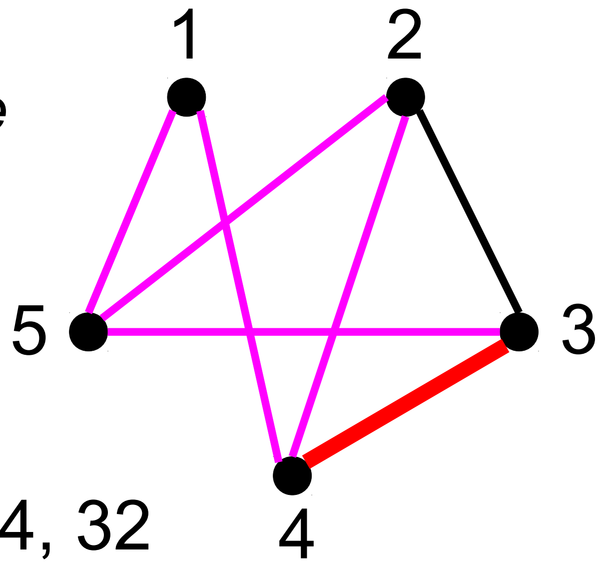

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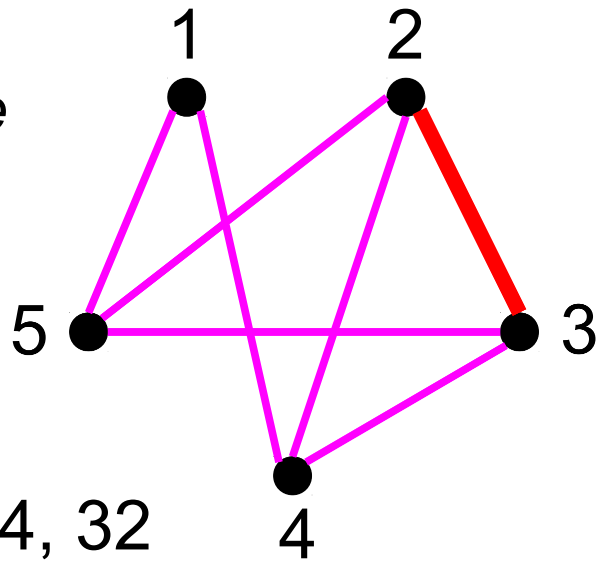

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Solitaire UNO as Hamiltonicity

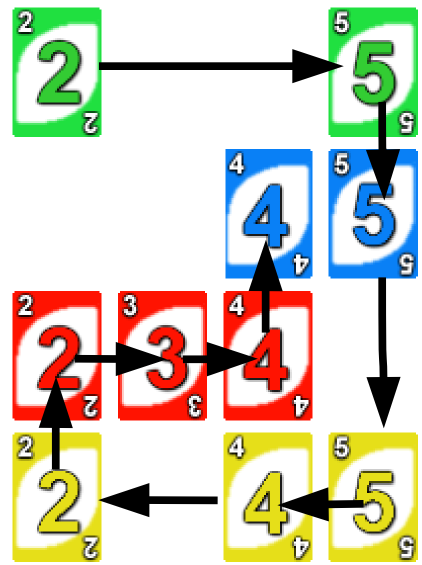
UNO – Rules:

Deck of cards
- c colors;
- b numbers.

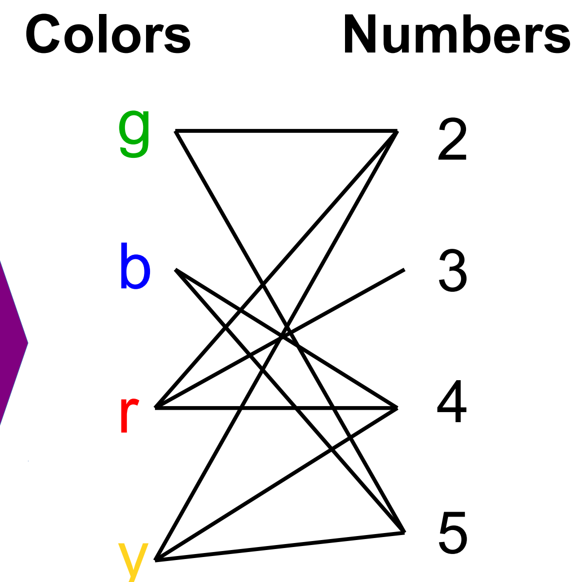
Matching rule: Cards agree either in *number* or in *color*.

- Given m cards, discard them one by one, following the matching rule.

EHP as Vertex Hamiltonicity



EHP as Edge Hamiltonicity



Path: 2, 5, 5, 5, 4, 2, 2, 3, 4, 4

Parameterized EHP

- **Parameterized Complexity:** Except from input *n*, we consider an additional variable which is called the parameter. Our goal is to confine the combinatorial explosion to the parameter.
- Demaine et. al. in 2014 studied EHP on bipartite graphs parameterized by the size of the smaller part *c*, and gave an algorithm that runs in $n^{O(c^2)}$ time.
- Question: Is EHP on bipartite graphs FPT? In other words, is there an algorithm that runs in $f(c) \cdot n^{O(1)}$?

Our Results

- EHP parameterized by the size of a vertex cover admits a cubic kernel (and thus it is FPT);

  (Independently shown by Dey et. al. in FUN 2014)

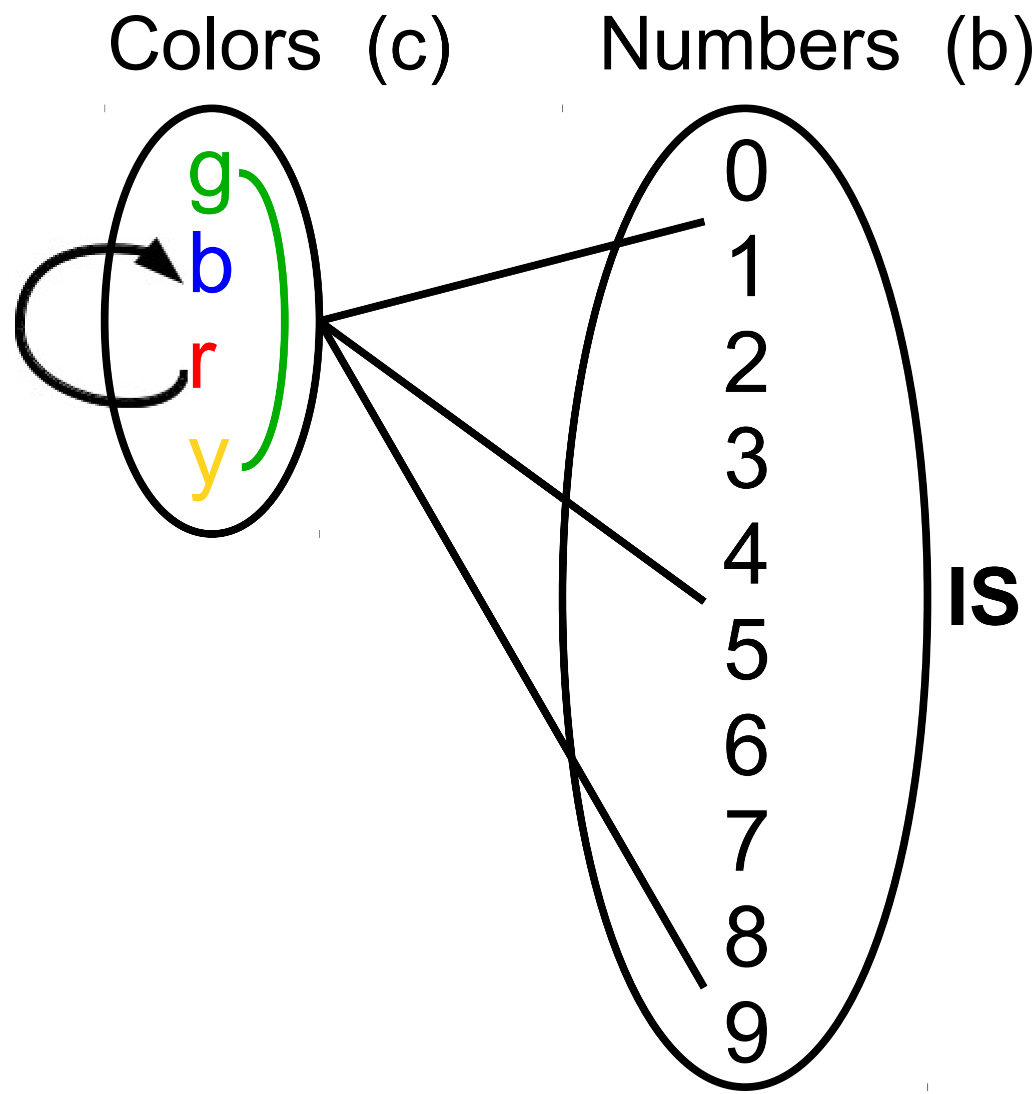
- EHP on arbitrary hypergraphs parameterized by the size of a hitting set is FPT;
- EHP parameterized by treewidth or cliquewidth is FPT.

  (Note: Vertex HP is FPT for tw but W-hard for cw.)

PART 1
EHP parameterized by VC and HS

Graphs of small Vertex Cover

- Graphs of small Vertex Cover are like bipartite graphs with one small part which might include edges. The large part is an independent set.
- All the edges are naturally associated with at least one color. By fixing an ordering in the colors, we can associate all edges with *exactly one color*.



Cubic Kernel for graphs of small VC

A kernel is an algorithm that runs in polynomial time and transforms an instance of a problem to one that depends only on the parameter.

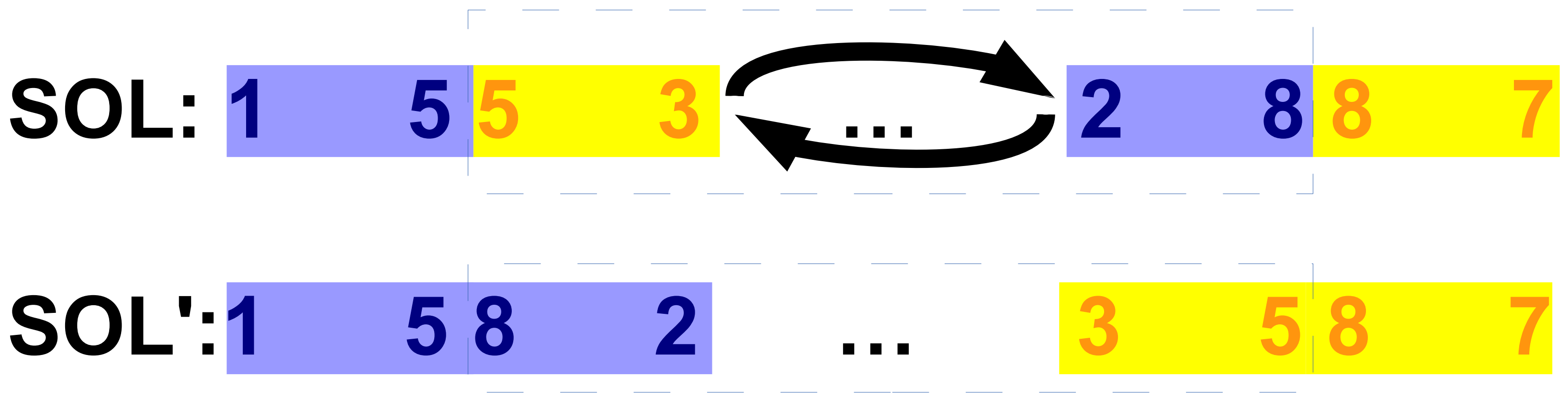
We will now describe a cubic kernel for graphs of small Vertex Cover.

Given an instance of EHP with c colors and b numbers, we construct an equivalent new instance that depends only on c:

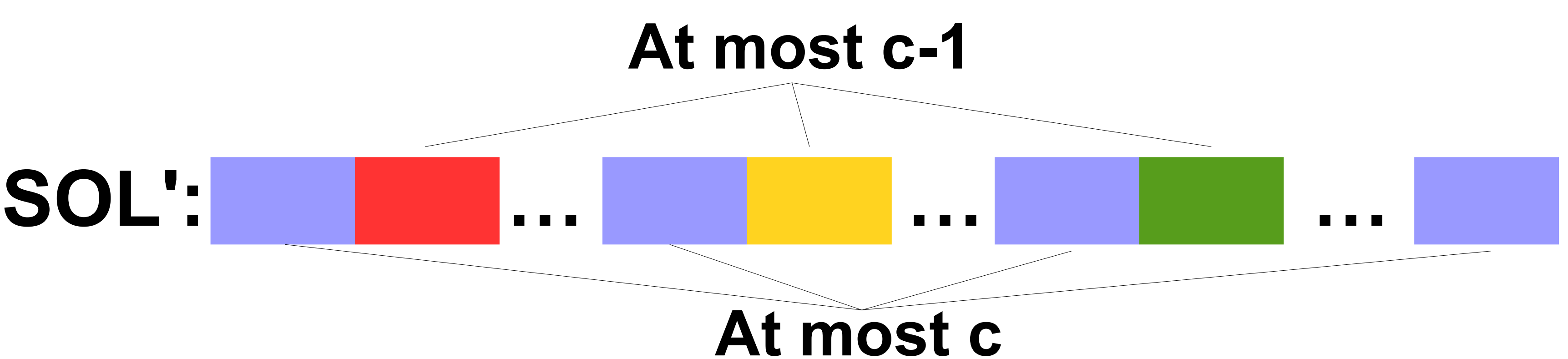
- For each color, there will be at most $O(c^2)$ numbers of that color
- Thus, there will be at most $O(c^3)$ numbers).

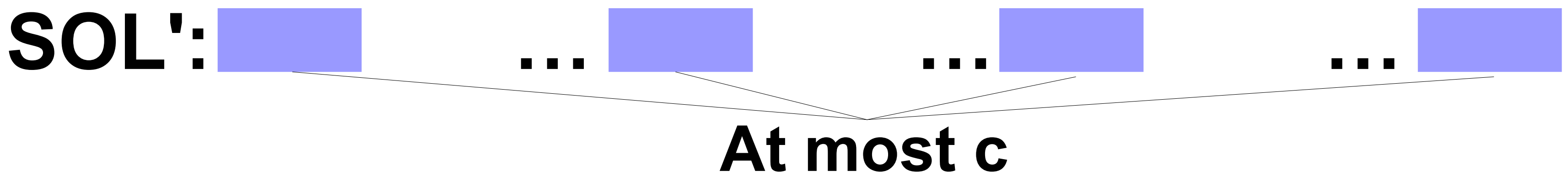
Given an EHP SOL, we construct a new EHP SOL', where each color-group will appear at most c times.

If two color groups appear one next to the other at least twice, then reverse the middle part of the EHP.

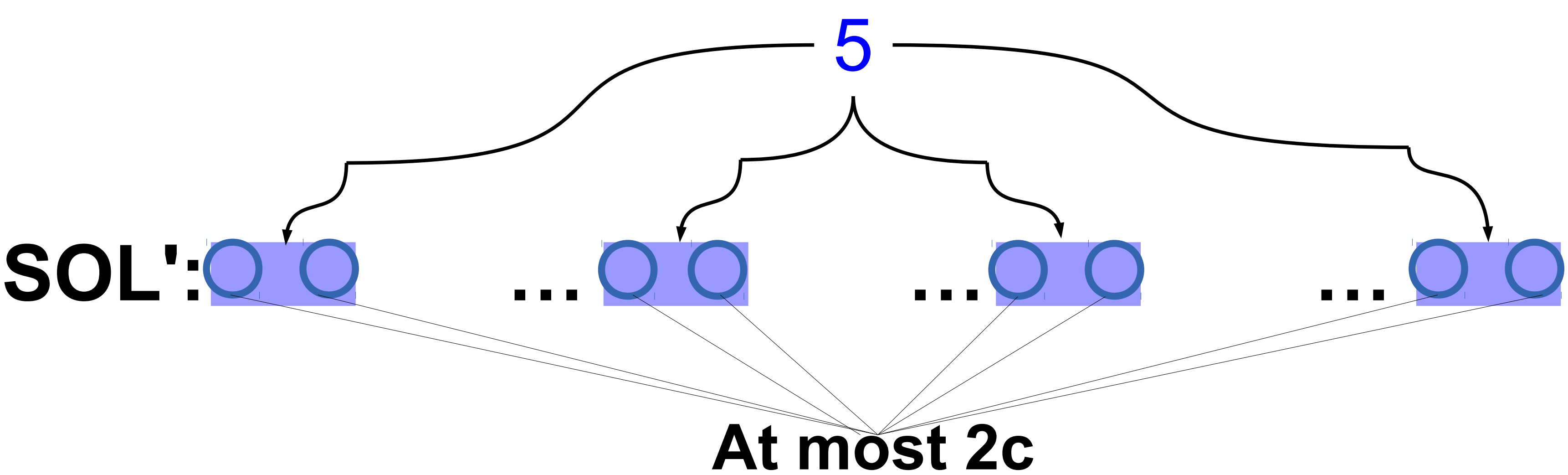


Now, all pairs of color groups should appear consecutively at most once.



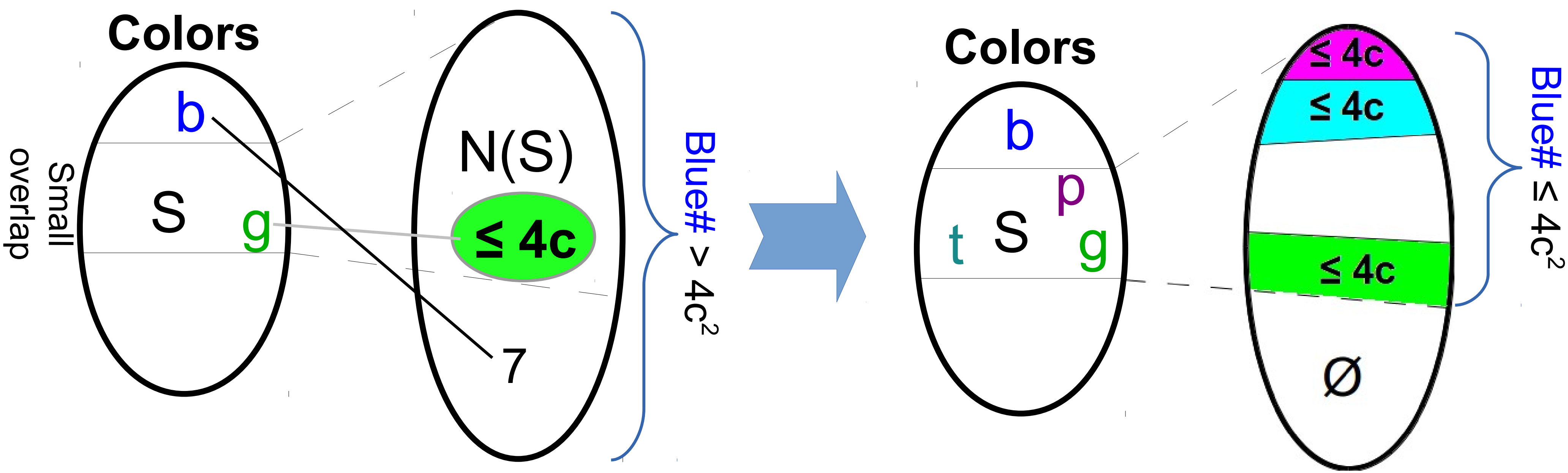


- Intuition: EHP has a small backbone, which we need to identify. All other edges can move freely in and out of the EHP.



- Bonus: What we have said so far works for Hypergraphs too.

- For graphs we can say something more clever: if a color has too many numbers, we can apply a reduction rule that removes some numbers of this color from the graph, without changing the solution.



Graph that contains 7 has an EHP **SOL** ⟺
Graph that doesn't contain 7 has an EHP **SOL'**.

PART 2
EHP parameterized by tw and cw

Treewidth and Cliquewidth

- Structural graph parameters:
  - Treewidth measures how tree-like a graph is.
  - Cliquewidth again measures graph complexity but is more general than tw (graphs of bounded tw also have bounded cw).

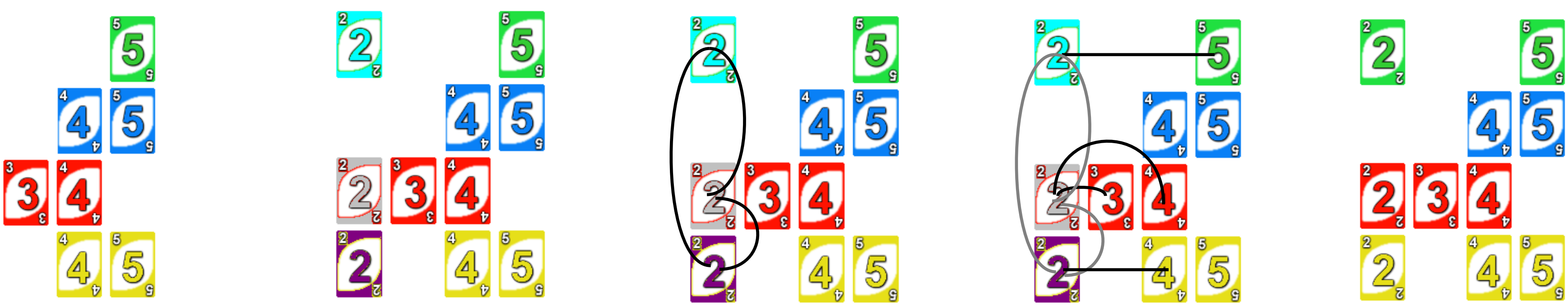
  tw is algorithmically more tractable than cw:
  - Problems expressible in $CMSO_2$ logic are tractable for graphs of bounded tw but not for graphs of bounded cw, ex. (vertex) Hamiltonian Path.

Definition of cliquewidth

A graph has cw k iff it can be constructed using k labels and the following operations:

- Introduce a new node with any label;
- Take the union of two graphs of cw k;
- Rename a label from i to j;
- Join the vertices of two labels i and j.

Example: UNO graphs have cliquewidth 2c.
<u>Inductively:</u>



Introduce a line of nodes using c new colors

Pairwise join new colors together;

Join color c+i to color i;

Rename color c+i to color i;

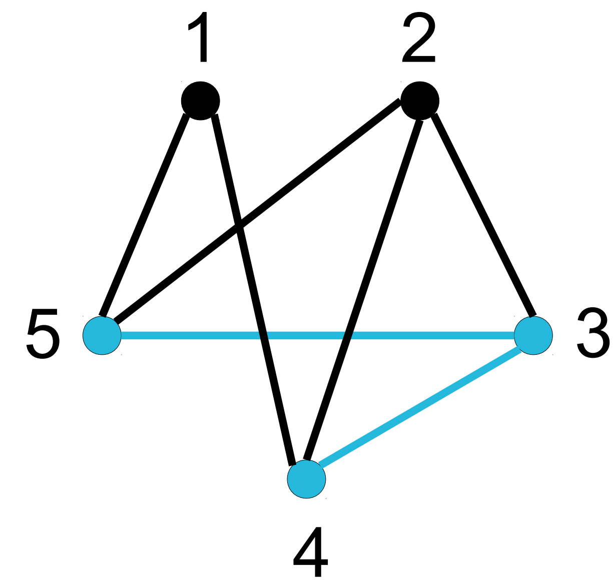
EHP as Dominating Eulerian Subgraph

Dominating Eulerian Subgraph

"Find a connected subgraph G' of G, st:

1. G' is Eulerian;

2. All remaining edges of G are covered by a vertex in G'."



Eulerian path: 53, 34
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Harary & Nash-Williams (1965): Edge-Hamiltonian Path is equivalent with Dominating Eulerian Subgraph.

Bonus: All the above properties are expressible in $CMSO_2$.

Dominating Eulerian Subgraph (and Edge-Hamiltonian Path) is FPT when parameterized by tw.

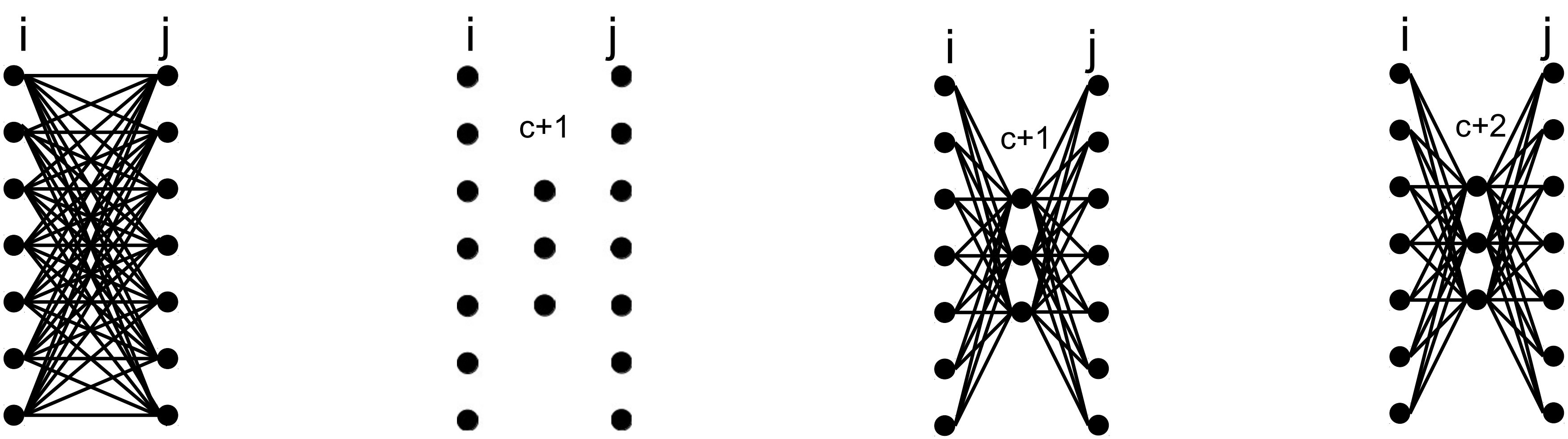
EHP (DES) parameterized by cw is FPT

Theorem (Gursky & Wanke 2000): If G has cw k and does not contain $K_{t,t}$ as a subgraph, then G has tw at most 3kt.

We can construct a $K_{t,t}$-free graph G' such that:

- G has Eulerian Dominating Subgraph iff G' does;
- If cw(G) = k, then cw(G') = k+2.

<u>Construction of G' - Idea:</u> If a $K_{t,t}$ is constructed in G by joining two large color-sets i and j, instead of joining do the following:



Construct 3 new vertices of label c+1;

Join i with c+1 and j with c+1;

Rename c+1 to c+2;