Complexity of generalized pleat folding

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Background story...

At 4th Japan Meeting on Origami in Science, Mathematics, and Education, 2008/6/22,

- Toshikazu Kawasaki, a mathematician and designer of Kawasaki rose, said that
  “For a mathematician, it is OK if solution exists.”

- A computer scientist, or I, cannot agree;
  - how to find the solution
  - the cost to find the solution
    - Goodness
    - Hardness

Is there any good problem just about computational cost?
• Pleat folding = 1D Origami

- Alternating foldings of Mountain and Valley
- Basic tool of Origami
- Many applications
  - Extension to General Patterns and consider its complexity...

Quite Important!!

Erik Demaine@5OSME
• Complexity of folding(?)

From the viewpoint of Computer Science

- Two resources of a computation model;
  1. time: the number of steps of operations
  2. space: the number of memory cells required to compute
• Complexity of folding(?)

From the viewpoint of Computer Science

- Two resources of Origami model?
  1. time...the number of *foldings* (operations)
  2. space...minimization of “*stretch*” that is
     - the number of papers between two hinged papers
New open problem

- **Least stretch folding problem**
  - **Input:** Paper of length $n+1$ and $s \in \{M, V\}^n$
  - **Output:** folded paper according to $s$
  - **Goal:** Find a *good* folded state with few *stretch*
    - At each crease, the number of papers between the papers hinged at the crease is *stretch*.
    - Two minimization problems;
      - minimize maximum
      - minimize total (=average)

It seems simple, ... so easy??  No!!
New open problem

- Simple non-trivial example

**Input:** MMVMMVMVVVV

The number of feasible folded states: **100**

**Goal:** Find a *good* folded state with few **stretch**

- The *unique* solution having min. max. value 3
  
  \[ [5|4|3|6|7|1|2|8|10|12|11|9] \]

- The *unique* solution having min. total value 11
  
  \[ [5|4|3|1|2|6|7|8|10|12|11|9] \]
New open problem

- Least **stretch** folding problem

**Input**: Paper of length \( n+1 \) and \( s \in \{M, V\}^n \)

**Output**: folded paper according to \( s \)

**Goal**: Find a **good** folded state with few **stretch**

- Two minimization problems; max/total(average)
- A few facts;
  - a pattern has a unique folded state iff it is pleats
  - solutions of \{min max\} and \{min total\} are different depending on a crease pattern.
  - there is a pattern having exponential combinations

\[ M \quad V \quad M \quad V \quad M \quad V \quad M \quad V \quad M \quad M \quad V \quad M \quad V \quad M \quad V \quad M \quad V \quad M \quad V \quad M \quad M \quad M \quad M \quad M \quad M \quad M \quad M \quad M \quad M \quad M \quad M \]

http://www.jaist.ac.jp/~uehara/etc/origami/
Least stretch folding problem

- Open problem:
  - Tractable/Intractable?
    - \( \mathcal{NP} \)-hard?
    - Poly-time solvable by Dynamic Programming?
    - ... up to now, I have no answer to this question ;-(

  - Then, exhaust search technique, that produces all possible folding ways, works?
    - average?

exponential pattern → pleats

No!!
Least stretch folding problem

- Partial answers;
  1. The number of folding ways for a random pattern
     - $\Theta(1.65^n)$ by experiments
     - $\Omega(1.53^n)$ and $O(2^n)$ by theoretical lower/upper bounds
       ... so a naïve program runs veeeeerrrrrrry slow.

[Note]
These results are based on enumeration & rough counting, described hereafter...
Least stretch folding problem

1. Average number of folding ways for a random pattern = \( f(n)/2^n \), where

\[
f(n) = \# \text{ of folding ways (or folded states) of a paper of length } n+1 \text{ (summation for all possible patterns)}
\]

- I give some bounds of \( f(n) \):
  - “The On-Line Encyclopedia of Integer Sequences” tells us up to \( n=28 \) (by enumeration);
    \( f(n) \sim \Theta(3.3^n) \).
  - I have upper/lower bounds;
    - upper bound: \( f(n) = O(4^n) \)
    - lower bound: \( f(n) = \Omega(3.07^n) \)
Least stretch folding problem

- Upper bound $f(n) = O(4^n)$ comes from the Catalan number.

[Proof] If the paper of length $n+1$ is folded, the crease points should be nested.

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Combination of $\frac{n}{2}$ pairs of () = Catalan Number $C_{\frac{n}{2}}$
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×
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Combination of $\frac{n}{2}$ pairs of () = Catalan Number $C_{\frac{n}{2}}$
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No way!!
Least stretch folding problem

- Lower bound \( f(n) = \Omega(3.07^n) \).

[Proof] We consider “folding of the last \( k+1 \) unit papers”;

We let
- \( f(n) \): the number of folding ways of length \( n+1 \)
- \( g(k) \): the number of folding ways of length \( k+1 \) s.t.
  the leftmost endpoint is not covered

Then, we have

\[
f(n) \geq (g(k))^{\frac{n}{k-1}} = \left( g(k)^{\frac{1}{k-1}} \right)^n
\]
Least stretch folding problem

- Lower bound $f(n) = \Omega(3.07^n)$.

[Proof] We consider “folding of the last $k+1$ unit papers”;

$g(k)$: the number of folding ways of length $k+1$ s.t. the leftmost endpoint is not covered

= “the number of ways a semi-infinite directed curve can cross a straight line $k$ times”,

A000682 in “The On-Line Encyclopedia of Integer Sequences”.

From that site, we have $g(43) = 830776205506531894760$.

Thus, by

$$f(n) \geq (g(k))^\frac{n}{k-1} = \left(g(k)^{1/(k-1)}\right)^n$$

we have the lower bound.
Open problem and Future work

- Least stretch folding problem:
  - Tractable/Intractable?
    - \( \mathcal{NP} \)-hard?
    - Poly-time solvable by Dynamic Programming?
    - Fixed parameter tractable for “stretch \(\leq k\)”?
  - What is the most complex pattern of M/V that has the most feasible folded states?

- Extension to...
  - non-unit length (operation-restricted linkage)
  - 2D (a kind of map folding)
  - What “space complexity” of Origami is?
  - area for folding?

Hiro Ito gives hardness proof?
Open problem and Future work

- Ext. of Least stretch folding problem:
  - non-unit length
    - First of all, there is a pattern that cannot be folded;
  - revisit unit length paper...
    - For any pattern, can it be folded?
      - “Yes”; by repeating “end-fold”
    - For any “folded state” of length 1, can it be folded?
      - If you allow *any folding*, it is “Yes” by
        - unlocked linkage in 2D ... reverse of bloom
        - universal theorem ... adding many creases
      - Is it “Yes” under simpler/reasonable model????
Universal theorem:

- Universal theorem of unit length folding in a simple folding model;
  - Input: a folded state of a paper strip of length $n+1$ to unit length
  - Question: Is it foldable on a suitable set of operations?
- Simple folding model

1. (from flat state)
2. pick up a point
3. valley fold most inner layers
4. to flat state

\[ \text{Diagram showing steps (a), (b), (c), (d).} \]
Universal theorem of linkage…

- Universal theorem of unit length folding in a simple folding model;
  Any flat folded state of unit length can be made by a sequence of simple foldings of length at most $2n$.

- Related results;
  - Any M/V pattern can be flat folded by repeating “end-folding”
  - If the creases are not unit-length, some folded state cannot be folded by simple foldings:
  - If “any folding” is allowed, “Yes” even in non-unit length
    - comes from “no locked linkage in 2D”!!
Universal theorem of linkage…

[Theorem] For a strip of paper, any unit-length flat folded state can be made by a sequence of simple foldings

[Proof]

Key idea 1: $P$ can be folded from a strip paper iff $P$ can be unfolded to flat strip paper by reversing. So we show that any folded state $P$ can be unfolded to flat by at most $2n$ simple unfoldings.

[Phase 1] unveil the endpoint
- unveil the covered endpoint and make it appear

[Phase 2] peal the paper from the endpoint
- unfold and extend the last flat part
Universal theorem of linkage…

[Proof]

[Phase 1] unveil the covered endpoint
- while we cannot see the endpoint, unfold the paper to expose the covered paper close to the endpoint.
- after at most $n$ unfoldings, we can see the endpoint.

[Phase 2] peal the paper from the endpoint
- unfold and extend the last flat part that contains the endpoint.
- after at most $n$ unfoldings, we obtain the flat paper.

[Note] At Phase 1, some new creases can be made by an “unfolding”. Hence the total number of “unfolding” cannot be bounded by $n$.
Universal theorem of linkage…

Open:
- The number of unfolding can be $n$?
- Characterization of “non-unfoldable” (non-unit length) origami by simple (un)folding.
- In simple folding model, no locked flat tree in 2D if edges are unit length?
Let’s turn to

the folding complexity of pleat folding

- Repeating of mountain and valley foldings
- Basic operation in some origami
- Many applications
Pleat folding

- Pleat folding (in 1D)

- Naïve algorithm: \( n \) time folding is a trivial solution
- We have to fold at least \( \log n \) times to make \( n \) creases

- More efficient ways…?
- General Mountain/Valley pattern?

- proposed at Open Problem Session on CCCG 2008 by R. Uehara.
• Complexity of Pleat Folding

[Main Motivation] Do we have to make \( n \) foldings to make a pleat folding with \( n \) creases??

1. The answer is “No”!
   - Any pattern can be made by \( \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \log n \right\rceil \) foldings

2. Can we make a pleat folding in \( o(n) \) foldings?
   - Yes!! ...it can be folded in \( O(\log^2 n) \) foldings.

3. Lower bound; \( \log n \)
   - \( \Omega(\log^2 n / \log \log n) \) lower bound for pleat folding!!
[Next Motivation] What about general pleat folding problem for a given M/V pattern of length $n$?

- **Any pattern can be made by** $\left\lfloor \frac{n}{2} \right\rfloor + \lceil \log n \rceil$ foldings
  1. Upper bound:
     Any M/V pattern can be folded by $(4+\varepsilon)\frac{n}{\log n} + o\left(\frac{n}{\log n}\right)$ foldings
  2. Lower bound:
     Almost all mountain/valley patterns require $\frac{n}{3+\log n}$ foldings

[Note] Ordinary pleat folding is exceptionally easy pattern!
Difficulty/Interest come from two kinds of Parities:
- “Face/back” determined by layers
- Stackable points having the same parity

**Input:** Paper of length $n+1$ and a string $s$ in $\{M, V\}^n$

**Output:** Well-creased paper according to $s$ at regular intervals.

**Basic operations**
1. Flat \{mountain-valley\} fold \{all/some\} papers at an integer point (= simple folding)
2. Unfold \{all/rewind/any\} crease points (= reverse of simple foldings)

**Rules**
1. Each crease point remembers the last folded direction
2. Paper is rigid except those crease points

**Goal:** Minimize the number of folding operations

**Note:** We ignore the cost of unfoldings
• Upper bound of **Unit FP** (1)

- Any pattern can be made by \( \lceil n/2 \rceil + \lceil \log n \rceil \) foldings
  1. *M/V* fold at center point according to the assignment
  2. Check the center point of the folded paper, and count the number of *Ms* and *Vs* (we have to take care that odd depth papers are reversed)
  3. *M/V* fold at center point taking majority
  4. Repeat steps 2 and 3
  5. Unfold all (cf. on any model)
  6. Fix all incorrect crease points one by one

**Steps 1~4 require \( \log n \) and step 6 requires \( n/2 \) foldings**
Upper bound of *Pleat Folding* \(^{(1)}\)

[Observation]
If \(f(n)\) foldings achieve \(n\) mountain foldings, \(n\) pleat foldings can be achieved by \(2f(n/2)\) foldings.

The following strategy works;
- Make \(f(n/2)\) mountain foldings at odd points;
- Reverse the paper;
- Make \(f(n/2)\) mountain foldings at even points.

We will consider the "*mountain folding problem*"
Mountain folding in $\log^2 n$ foldings

Step 1;
1. Fold in half until it becomes of length [vvv] (log $n-2$ foldings)
2. Mountain fold 3 times and obtain [MMM]
3. Unfold; vMMMvvvvvMMMVvvvvvMMMVvvvvvMMMVvvvvv...

Step 2;
1. Fold in half until all “vvvvv”s are piled up (log $n-3$ foldings)
2. Mountain fold 5 times [MMMMMMM], and unfold
3. vMvMMMMMMMMvMvvvvvMvMMMMMMMMvMvvvvvMvM

Step 3; Repeat step 2 until just one “vvvvv” remains
vMvMMMMvvMMMMMvMMMMMvMMMMvMvMvM

Step 4; Mountain fold all irregular vs step by step.

- #iterations of Steps 2~3; $\log n$
- #valleys at step 4; $\log n$
- #foldings in total~ $(\log n)^2$
• **Lower bound of Unit FP**

[Thm] Almost all patterns but o(2^n) exceptions require \( \Omega(n/\log n) \) foldings.

[Proof] A simple counting argument:

- # patterns with \( n \) creases > 2^n/4 = 2^{n-2}
- # patterns after \( k \) foldings < \((2 \times n) \times (n+1) \times (2 \times n) \times (n+1) \times \ldots \times (n+1) \times (2 \times n) < (2n(n+1))^k\)
- We cannot fold most patterns after at most \( k \) foldings if \( \sum_{i=0}^{k} (2n(n+1))^i \leq (2n(n+1)+1)^k < 2^{n-2} \)
- Letting \( n \geq 2, k = O\left(\frac{n}{\log n}\right) \) we have \((2n(n+1)+1)^k = o(2^n)\)
Any pattern can be folded in $cn/\log n$ foldings

Prelim.
- Split into *chunks* of size $b$;
  1. Each chunk is small and easy to fold
  2. #kinds of different $b$s are not so big

Main alg.
- For each possible $b$
  1. pile the chunks of pattern $b$ and mountain fold them
  2. fix the reverse chunks
  3. fix the boundaries

Select suitable $b$ depending on $n$.  

Half chunks are done!  

Fold NOK $s$ again

Repeat for all chunks

Analysis is omitted
• Open Problems

- Pleat foldings
  - Make upper bound $O(\log^2 n)$ and lower bound $\Omega(\log^2 n / \log \log n)$ closer

- “Almost all patterns are difficult”, but…
  - No explicit M/V pattern that requires $(cn / \log n)$ foldings

- When “unfolding cost” is counted in…
  - Minimize #foldings + #unfoldings